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# BFKL at Next-to-Next-to-Leading Order

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**Abstract**

We determine an approximate expression for the $O(\alpha_s^3)$ contribution $\chi_2$ to the kernel of the BFKL equation, which includes all collinear and anticollinear singular contributions. This is derived using recent results on the relation between the GLAP and BFKL kernels (including running-coupling effects to all orders) and on small-$x$ factorization schemes. We present the result in various schemes, relevant both for applications to the BFKL equation and to small-$x$ evolution of parton distributions.

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## 1. The BFKL equation beyond next-to-leading order

Higher order calculations in perturbative QCD, both at fixed order and at the resummed level, are playing an increasingly important role in precision collider physics [1]. Fixed-order and resummed results pose important constraints on each other. On the one hand, fixed-order computations determine an infinite number of coefficients of resummed expressions: in the soft (e.g. large-$x$) limit [2,3], the resummed hard cross-section can be determined to any desired logarithmic order by a finite fixed-order computation of suitably high accuracy, while at high energy (e.g. small $x$) there is a duality [4] such that a finite-order computation of the GLAP kernel determines an all-order resummation of the BFKL kernel and conversely. On the other hand, available resummed results determine partly higher fixed-order expressions and allow an approximate reconstruction of their form: in fact, approximate determinations of the next-to-next-to-leading order GLAP splitting functions [5], which played a useful role phenomenological until the exact expressions became available[6], are significantly constrained by knowledge of the large-$x$ and small-$x$ behaviour from resummed results.

The interplay of finite-order and resummed results is especially interesting for the high-energy (small-$x$) limit of hard cross-sections, which behave as genuine two-scale processes. The dependence on the hard scale (henceforth $Q^2$, for definiteness) and the energy scale (actually, the dimensionless ratio, $x$ for definiteness, that controls the energy dependence) are governed by a pair of evolution equations, the GLAP and BFKL equations, respectively, whose kernels are related by a duality relation which was recently shown to hold to all orders at the running–coupling level [7]. This duality relation determines the resummed expansion of either kernel in terms of the fixed-order expansion of the other. Hence, even when the symmetry between the two scales is broken by the running of the coupling (and/or by kinematics) knowledge of either of the two kernels at fixed orders enables an all-order resummation of the other kernel.

These results have been mainly used to perform small-$x$ resummation of GLAP evolution (see refs. [8,9] and refs. therein, and refs. [10,11] for phenomenological applications), i.e. to learn about higher-order contributions to the GLAP kernel. However, they can also be used to determine higher-order contributions to the BFKL kernel: indeed, they provided a nontrivial check on the next-to-leading order determination of the BFKL kernel [12,13]. Their use to determine corrections to the BFKL kernel beyond next-to-leading order, which are hitherto unknown, has been hampered by two difficulties. First, whereas some next-to-next-to leading order duality relations have been worked out some time ago [14], they didn't include the full next-to-leading order running of the coupling, nor the information contained in the then unknown NNLO GLAP kernel. In fact, the inclusion of these contributions is quite hard if the running–coupling duality relations are solved by brute force, even using computer algebra as in Ref. [14]. Second, beyond leading order both fixed-order and resummed results, or, equivalently, both the BFKL kernel and the GLAP kernel depend on a choice of factorization scheme. Factorization schemes at small-$x$ were determined explicitly in refs. [15,16] and their implications for the GLAP-BFKL duality were worked out in refs. [17,10], but only up to next-to-leading order.

Recent results solve both difficulties. In ref. [7] a general method has been developed which allows an efficient determination of duality relations by purely algebraic techniques,

with full inclusion of the running of the coupling to any desired order. Thanks to the full computation of the GLAP splitting functions to next-to-next-to leading order, [6] a computation of all the singular contributions to the next-to-next-to leading order BFKL kernel $\chi_2$ is now possible. Furthemore in ref. [18] small-$x$ scheme changes have been discussed to all orders. Even though in ref. [18] the scheme change required for the BFKL-GLAP is determined explicitly only up to NLO, it turns out that its determination up to NNLO is possible from available results, as we shall discuss below, at least for the terms which affect the singular contributions to the NNLO BFKL kernel. The purpose of the present paper is to perform these computations, and use them to construct an approximate form of the $\chi_2$ kernel.

A determination of the BFKL kernel to NNLO is of theoretical and phenomenological interest for various reasons. It is well-known that NLO corrections to the BFKL kernel are large and in fact change completely its qualitative shape. The determination of the NNLO contribution is thus motivated not only by the slow convergence of the perturbative expansion of the BFKL kernel, but also by the expectation that (because of the alternating sign of the dominant contributions) the NNLO approximation has a minimum like the LO, and thus better stability properties than the NLO. Also, it is unclear whether a direct extraction of the NNLO BFKL kernel from the high-energy behaviour of parton-parton scattering amplitudes analogous to the NLO computation of Ref. [12] is feasible, because it is unclear whether beyond NLO some form of "reggeization" holds, i.e. whether the exchange of an effective colorless multigluon state is universal [19,20]. If this is not the case, a derivation of the NNLO BFKL kernel from high-energy factorization may be the only viable option.

This paper will be organized as follows. In the next section we will summarize the formalism of Ref. [7] for the algebraic resolution of duality relations between the GLAP and BFKL kernel, and describe specifically its application to the extraction of all available information on the NNLO BFKL kernel from the known NNLO GLAP result. Then in Section 3 we will turn to the issue of scheme dependence at small $x$ at NNLO: after summarizing the general results of ref. [18], we will describe its application to NNLO duality. In Section 4 we will finally determine explicitly the approximate NNLO $\chi_2$ kernel in various relevant factorization schemes, discuss its features and estimate the accuracy of our approximation. Technical results on higher-order dualities and on the so-called $Q_0$ scheme at NNLO are collected in the appendices.

## 2. The GLAP–BFKL duality

Duality is the statement that the solutions to the GLAP and BFKL equations coincide up to higher-twist corrections provided their kernels are suitably matched. Its consequence is that the leading-twist part of each kernel is determined by the other kernel. This result is straightforward to establish at fixed coupling [21,4,22], but rather more subtle when the coupling runs [23,7]. Here we summarize the main results, while referring to Ref. [7] for a more comprehensive treatment.

We discuss evolution equations for a parton distribution $G(x, Q^2)$, which can be thought of simply as the gluon density, or as an eigenvector of a two-by-two evolution

matrix in the singlet sector. The kinematic variables $x$ and $Q^2$ can be thought of as the standard DIS variables, or more generally the perturbative scale and a dimensionless scale ratio such that $0 \leq x \leq 1$. We will not consider the dependence on other kinematic variables, such as transverse momentum and rapidity, i.e. we will consider standard parton distributions, for which ordinary collinear factorization applies. We will therefore limit ourselves to angular-averaged quantities at the leading-twist level. We express the parton density as a function of the logs of the relevant kinematic variables:

$$G = G(\xi, t); \qquad \xi \equiv \log\frac{1}{x}, \quad t \equiv \log\frac{Q^2}{\mu^2}, \tag{2.1}$$

and define the Mellin transform with respect to either (or both) of the kinematic variables:

$$G(N,t) \equiv \int_0^\infty d\xi\, e^{-N\xi} G(\xi,t), \tag{2.2}$$

$$G(\xi,M) \equiv \int_{-\infty}^\infty dt\, e^{-Mt} G(\xi,t). \tag{2.3}$$

Note that, by slight abuse of notation, we denote with the same symbol the parton distributions $G(N,t)$, $G(\xi,M)$, and $G(N,M)$, although they are of course different functions of the respective arguments.

The GLAP equation and BFKL equations express respectively the $t$ or $\xi$ dependence of $G$. They take the form

$$\frac{d}{dt}G(N,t) = \gamma(\alpha_s(t),N)G(N,t), \tag{2.4}$$

$$\frac{d}{d\xi}G(\xi,M) = \chi(\hat\alpha_s,M)G(\xi,M), \tag{2.5}$$

where $\alpha_s(t)$ is the running coupling, which upon Mellin transformation becomes the operator $\hat\alpha_s$, obtained by replacing $t \to -\frac{\partial}{\partial M}$ in the expression for $\alpha_s(t)$. For example, at the leading-log level, where $\alpha_s(t) = \alpha_s/(1+\beta_0\alpha_s t)$,

$$\widehat\alpha_s = \frac{\alpha_s}{1-\beta_0\alpha_s\frac{\partial}{\partial M}}, \tag{2.6}$$

with $\alpha_s \equiv \alpha_s(0)$.

At *fixed* coupling it is easy to show that if the kernels $\chi$ and $\gamma$ are related by

$$N = \chi(\alpha_s, \gamma(\alpha_s, N)), \tag{2.7}$$

$$M = \gamma(\alpha_s, \chi(\alpha_s, M)), \tag{2.8}$$

then the BFKL and GLAP equation admit the same solution. This relation is straightforward to derive [24,22] from the observation that the leading-twist behaviour of $G(N,M)$ is determined by its pole in the $(M,N)$ plane, and that eqs. (2.7) and (2.8) express the

position of this pole. This duality maps the expansion of $\chi(\alpha_s, M)$ in powers of $\alpha_s$ at fixed $M$ onto the expansion of $\gamma(\alpha_s, N)$ in powers of $\alpha_s$ at fixed $\alpha_s/N$, and the expansion of $\gamma(\alpha_s, N)$ in powers of $\alpha_s$ at fixed $N$ onto the expansion of $\chi(\alpha_s, M)$ in powers of $\alpha_s$ at fixed $\alpha_s/M$.

At the running–coupling level, duality relations can be derived order by order by solving eqs. (2.4) and (2.5) perturbatively and matching the respective solutions [14]. That this is possible beyond next-to-leading order is highly nontrivial, and it can be proven to all orders using operator methods [7]. Namely, one shows that at the running–coupling level, the BFKL and GLAP solutions coincide if the respective kernels, viewed as operators in $(M, N)$ space, are the inverse of each other when acting on physical states, i.e. such that if

$$MG(N, M) = \gamma(\hat{\alpha}_s, N)G(N, M), \tag{2.9}$$

then

$$NG(N, M) = \chi(\hat{\alpha}_s, M)G(N, M), \tag{2.10}$$

and conversely. Note that the conditions eq. (2.9) and (2.10) should be viewed as conditions on the action of the operators $\gamma(\hat{\alpha}_s, N)$ and $\chi(\hat{\alpha}_s, M)$ on physical states: specifically, they are not just the Mellin transforms of eqs. (2.4),(2.5), which depend on a boundary condition, but rather, they express a property of the leading–twist Green function of perturbative GLAP or BFKL evolution [7].

The inversion is nontrivial because the operators involved do not commute. Indeed, we can start with eq. (2.9) and construct $\tilde{\chi}(\hat{\alpha}_s, M)$ as a series in $M$ such that

$$\tilde{\chi}(\hat{\alpha}_s, \gamma(\hat{\alpha}_s, N))G(N, M) = NG(N, M). \tag{2.11}$$

Because $\hat{\alpha}_s$ and $N$ commute, $\tilde{\chi}(\hat{\alpha}_s, M)$ coincides with the inverse function of $\gamma$ i.e. the fixed-coupling (or naive) dual eq. (2.7): it is the same power series in $M$. However, because $\gamma(\hat{\alpha}_s, N)$ and $M$ do *not* commute, we cannot use eq. (2.9) to identify $\gamma(\hat{\alpha}_s, N)$ with $M$ in eq. (2.11), and thus obtain the desired equation (2.10).

The problem can be formalized as follows: given an operator equation of the form

$$\hat{q}G = \hat{p}G, \tag{2.12}$$

and given a function $f(\hat{q})$, determine the function $g$ such that using eq. (2.12) one gets

$$f(\hat{q})G = g(\hat{p})G. \tag{2.13}$$

It is easy to see that when $\hat{p}$ and $\hat{q}$ do not commute, the functions $f$ and $g$ do not coincide: in fact they can be determined in terms of each other using the expansion of the Baker-Campbell-Hausdorff formula [25] to lowest nontrivial order:

$$f(\hat{q})G = \left\{f(\hat{p}) - \tfrac{1}{2}f''(\hat{p})\,[\hat{p}, \hat{q}] + \dots\right\}G, \tag{2.14}$$

so that

$$g(\hat{p}) = f(\hat{p}) - \tfrac{1}{2}f''(\hat{p})\,[\hat{p}, \hat{q}] + \dots. \tag{2.15}$$

Expressions up to third order are derived in ref. [7], and higher-order expressions (up to fifth order) are given in Appendix A.

Hence, if we identify

$$\hat{p} = M, \qquad \hat{q} = \gamma(\hat{\alpha}_s, N), \tag{2.16}$$

and then choose $f = \tilde{\chi}$, we may use eq. (2.15) (and its higher-order generalizations) to determine $\tilde{\chi}(\hat{\alpha}_s, \gamma(\hat{\alpha}_s, N))$ on the l.h.s. of eq. (2.11) as a function of $M$, which we then identify with the sought-for BFKL kernel $\chi(\hat{\alpha}_s, M)$. The computation of $\chi(\hat{\alpha}_s, M)$ is thus reduced by eq. (2.14) to the determination of commutators, which in our case can all be obtained from the basic commutator

$$[\hat{\alpha}_s^{-1}, M] = -\beta_0 + \alpha_s \beta_0 \beta_1 + \ldots, \tag{2.17}$$

where the QCD beta function is given by

$$\beta(\alpha_s) = -\beta_0 \alpha_s^2 (1 + \beta_1 \alpha_s) + \ldots. \tag{2.18}$$

Using this commutator in eq. (2.14), to lowest nontrivial order we get

$$NG(N, M) = \left[\tilde{\chi}(\hat{\alpha}_s, M) + \frac{1}{2}\beta_0 \hat{\alpha}_s^2 \frac{\partial}{\partial \hat{\alpha}_s} \gamma(\hat{\alpha}_s, N)\, \tilde{\chi}''(\hat{\alpha}_s, M)\right] G(N, M) + O(\hat{\alpha}_s^2), \tag{2.19}$$

where primes denote derivatives with respect to the second argument ($M$ or $N$).

In order for the r.h.s. of eq. (2.19) to provide us with an expression for $\chi(\hat{\alpha}_s, M)$ we must eliminate the residual $N$ dependence in it. This can be done in two steps. First, we can back-substitute the lower-order expressions for $N$ given by eq. (2.19) in its higher-order terms: for example, eq. (2.19) tells us that we can just replace the leading order expression $\tilde{\chi}(\hat{\alpha}_s, M)$ for the $N$ dependence of the next-to-leading order $O(\beta_0)$ term, up to $O(\hat{\alpha}_s^2)$ corrections:

$$NG(N, M) = \left[\tilde{\chi}(\hat{\alpha}_s, M) + \frac{1}{2}\beta_0 \hat{\alpha}_s^2 \frac{\partial}{\partial \hat{\alpha}_s} \gamma(\hat{\alpha}_s, \tilde{\chi}(\hat{\alpha}_s, M))\, \tilde{\chi}''(\hat{\alpha}_s, M)\right] G(N, M) + O(\hat{\alpha}_s^2). \tag{2.20}$$

Of course, beyond next-to-leading order this back-substitution becomes nontrivial, and it must be performed by keeping into account the commutation properties of the operators involved.

The dependence of the result eq. (2.20) on $\gamma(\hat{\alpha}_s, \tilde{\chi}(\hat{\alpha}_s, M))$ and its derivatives can be finally expressed using the duality relation eq. (2.8) in terms of $M$ and of derivatives of the fixed-coupling dual $\tilde{\chi}$ of $\gamma$: e.g. differentiating eq. (2.8) with respect to $\alpha_s$ or with respect to $M$ gives respectively

$$\frac{\partial}{\partial \alpha_s} \gamma(\alpha_s, \tilde{\chi}(\alpha_s, M)) + \gamma'(\alpha_s, \tilde{\chi}(\alpha_s, M)) \frac{\partial}{\partial \alpha_s} \tilde{\chi}(\alpha_s, M) = 0, \tag{2.21}$$

$$\gamma'(\alpha_s, \tilde{\chi}(\alpha_s, M)) \tilde{\chi}'(\alpha_s, M) = 1, \tag{2.22}$$

and so on. Using this, the right-hand side of eq. (2.20) gives us finally an expression for $\chi(\hat{\alpha}_s, M)$ in terms of $\tilde{\chi}(\hat{\alpha}_s, M)$, i.e. for the running–coupling dual BFKL kernel in terms of

the naive dual one, which in turn can be determined from the anomalous dimension $\gamma$ using fixed-coupling duality eq. (2.7). To lowest nontrivial order this gives the well-known [26,24] result

$$\chi(\hat{\alpha}_s, M) = \tilde{\chi}(\hat{\alpha}_s, M) - \frac{1}{2}\beta_0\hat{\alpha}_s\tilde{\chi}(\hat{\alpha}_s, M)\frac{\tilde{\chi}''(\hat{\alpha}_s, M)}{\tilde{\chi}'(\hat{\alpha}_s, M)} + O(\hat{\alpha}_s^2), \tag{2.23}$$

where we made use of the fact that the kernels start at order $\hat{\alpha}_s$, so $\partial\chi(\hat{\alpha}_s, M)/\partial\hat{\alpha}_s = \hat{\alpha}_s^{-1}\chi(\hat{\alpha}_s, M) + O(\hat{\alpha}_s)$. Equation (2.23) expresses the BFKL kernel $\chi$ which is dual to the given GLAP anomalous dimension $\gamma$, in terms of the fixed-coupling dual $\tilde{\chi}$ determined from $\gamma$ using eq. (2.7).

Analogously, one can construct the GLAP anomalous dimension $\gamma$ which is dual to a given $\chi$ in terms of the fixed-coupling dual $\tilde{\gamma}$ determined from $\chi$ using eq. (2.8):

$$\gamma(\hat{\alpha}_s, N) = \tilde{\gamma}(\hat{\alpha}_s, N) - \frac{1}{2}\beta_0\hat{\alpha}_s\chi(\hat{\alpha}_s, \tilde{\gamma}(\hat{\alpha}_s, N))\frac{\chi''(\hat{\alpha}_s, \tilde{\gamma}(\hat{\alpha}_s, N))}{[\chi'(\hat{\alpha}_s, \tilde{\gamma}(\hat{\alpha}_s, N))]^2} + O(\hat{\alpha}_s^2), \tag{2.24}$$

where $\chi(\hat{\alpha}_s, \tilde{\gamma}(\hat{\alpha}_s, N))$ can be further expressed in terms of $\tilde{\gamma}(\hat{\alpha}_s, N)$ and its derivatives differentiating the duality relation between $\chi$ and $\tilde{\gamma}$, analogously to eqs. (2.21),(2.22). It is important to note that $\tilde{\gamma}$ and $\tilde{\chi}$ are not the fixed-coupling dual of each other: rather each of them is the fixed-coupling dual of (respectively) $\chi$ and $\gamma$, which are related by running–coupling duality. It is sometimes convenient [8] (and it will be useful for our discussion of scheme changes) to view $\tilde{\chi}$ as an effective $\chi$ kernel: namely, once $\gamma$ is determined from a given $\chi$ using eq. (2.23), we also define $\tilde{\chi}$ which is the fixed-coupling dual of this $\gamma$ eq. (2.23).

In ref. [7] we have used this approach to derive the running–coupling corrections to the small-$x$ resummation of the GLAP kernel which one obtains from BFKL, i.e. essentially eq. (2.24), up to next-to-next-to-leading order. In the sequel of this paper, we will use it to derive the running coupling contributions to the next-to-next-to leading order BFKL kernel obtained starting from the GLAP kernel, i.e. essentially eq. (2.23). Various higher-order running–coupling duality relations are derived and collected in Appendix A.

Because running–coupling corrections to duality are given as a series in $\alpha_s$ of terms each of which is a function of the fixed-coupling dual expression, also at the running–coupling level duality maps the expansion of $\chi(\alpha_s, M)$ in powers of $\alpha_s$ at fixed $M$ onto the expansion of $\gamma(\alpha_s, N)$ in powers of $\alpha_s$ at fixed $\alpha_s/N$, and conversely. This means that knowledge of $\gamma$ up to next-to-next-to leading order allows us to determine all collinear ($M \sim 0$) singular contributions to $\chi$ up to next-to-next-to leading level. Specifically, if we expand

$$\chi(\hat{\alpha}_s, M) = \hat{\alpha}_s\chi_0(M) + \hat{\alpha}_s^2\chi_1(M) + \ldots, \tag{2.25}$$

and then

$$\chi_i(M) = \frac{c_{i,-i-1}}{M^{i+1}} + \frac{c_{i,-i}}{M^i} + \ldots, \tag{2.26}$$

for some coefficients $c_{i,j}$, we can determine the first three orders of the expansion eq. (2.26) of $\chi_i(M)$ for all $i$, i.e. $c_{i,j}$ for all $i$ and $j = -i-1, -i, -i+1$. Furthermore, the symmetry properties of $\chi$ allow us to determine its expansion about $M = 1$ from knowledge of the coefficients of the expansion about $M = 0$. This procedure requires some care in the

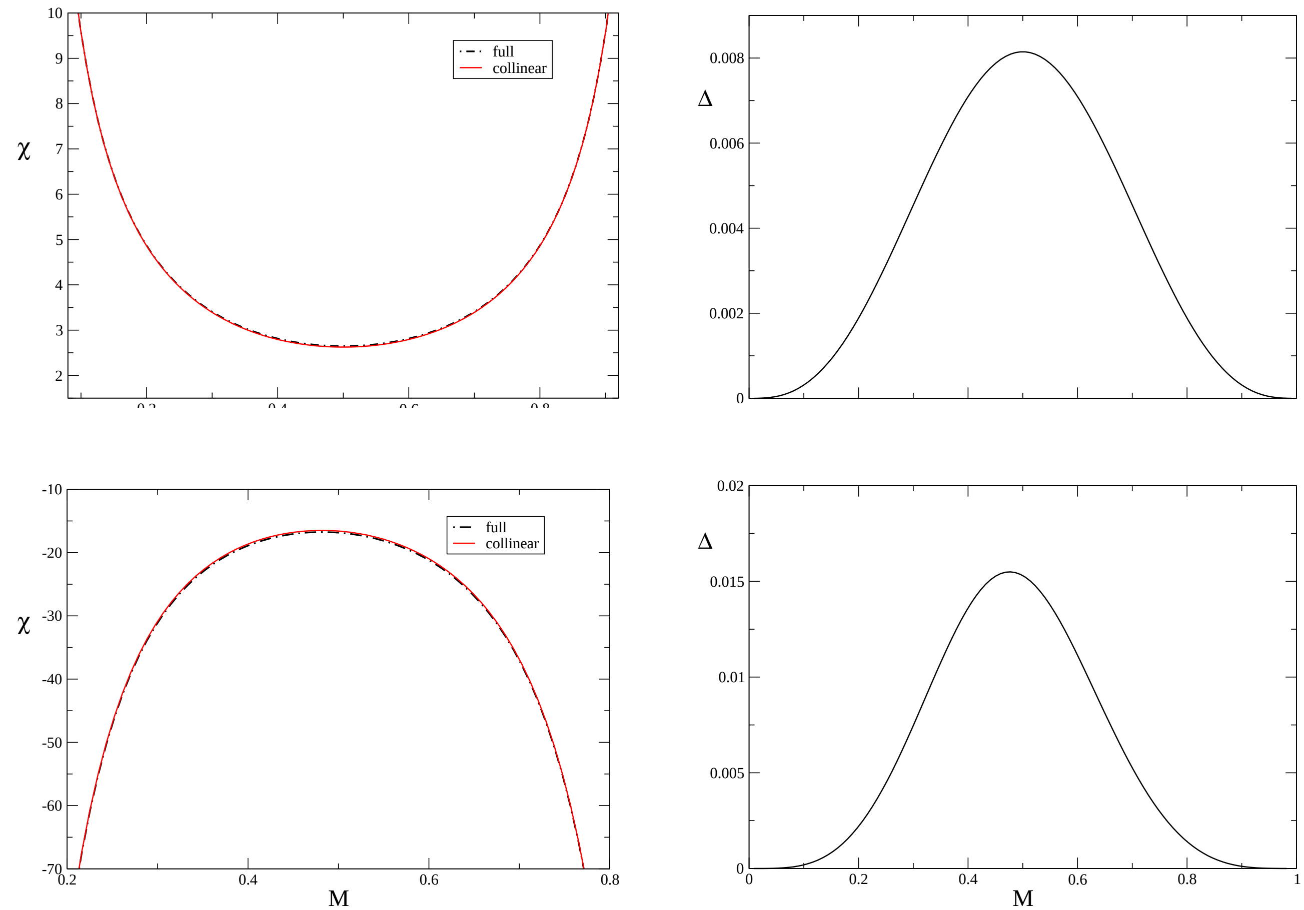


Figure 1: Comparison of the exact and approximate expressions of the leading–order and next-to-leading–order contributions to the BFKL kernels $\chi_0(M)$ and $\chi_1(M)$. The approximate expressions (given explicitly in appendix D) are obtained by symmetrizing the third-order collinear expansion about $M = 0$ of the kernel, as discussed in section 4. The upper two plots are the leading-order $\chi_0$, while the lower are the next-to-leading-order $\chi_1$: in each case on the left we compare the exact (dot-dash) and approximate (solid red) expressions as a function of $M$, while on the right we show the relative difference (exact − approximate)/exact. See section 4 for a full discussion of the construction of the approximation and its scheme dependence.

treatment of the running of the coupling, which affects the way the symmetry is realized, and it will be discussed in sect. 4. In the specific case of $\chi_2$, knowledge of $\gamma$ up to NNLO allows us thus to determine all the singular contributions to $\chi_2$ at $M = 0$ and $M = 1$, i.e. all the collinear and "anticollinear" singularities respectively.

Because $\chi_0$ and $\chi_1$ are known exactly, we can test the accuracy of the approximation to them which is obtained by retaining only the first three terms of their expansion about $M = 0$ and $M = 1$ (and subtracting the double-counting between the two expansions). This comparison is displayed in figure 1 for the leading and next-to-leading order contributions $\chi_0$ and $\chi_1$ to the BFKL kernel, where it is seen that the approximation is exceedingly good. The percentage accuracy of the approximate expression is better than 0.8% at leading order and better than 1.5% at next-to-leading order.

This motivates us to consider the construction of a similar approximation to $\chi_2$. Before doing this, however, we must turn to issues of factorization scheme dependence.

## 3. Factorization schemes at small $x$

Running–coupling duality, reviewed in the previous section, implies that, up to higher-twist corrections, the solution to a GLAP-like equation (2.4) can also be obtained as the solution to a BFKL-like equation (2.5), and it tells us how the evolution kernels of the two equations are related. However, as is well known, beyond leading order, both the GLAP and BFKL kernels are only defined up to a choice of factorization scheme. Namely, if the normalization of $G(\xi, t)$ is redefined by a subleading function $Z(\alpha_s, N) = 1 + O(\alpha_s)$, the evolution kernel beyond leading order changes: hence, the kernel is uniquely defined only once the normalization is fixed.

In general, of course, factorization schemes can mix the singlet quark and gluon with each other. Here, however, we will only consider factorization-scheme changes of the single parton distribution $G(\xi, t)$ which enters both the GLAP and BFKL equations: because only one eigenvector of the anomalous dimension matrix has leading small-$x$ singularities, the BFKL equation is one-dimensional, and mixing is irrelevant for duality. The general matching of scheme changes which have effect at small $x$ with those which do involve operator mixing (relevant for phenomenology) is discussed in detail in refs. [17,10]. The most general factorization-scheme change is then obtained by redefining

$$G'(t, N) = Z_{sx}\left(\alpha_s, \frac{\alpha_s}{N}\right) Z_{lx}\left(\alpha_s, N\right) G(t, N), \tag{3.1}$$

where the large-$x$ and small-$x$ scheme change functions have respectively the form

$$Z_{lx}\left(\alpha_s, N\right) = 1 + \alpha_s Z^1_{lx}(N) + O(\alpha_s^2) \tag{3.2}$$

$$Z_{sx}\left(\alpha_s, \frac{\alpha_s}{N}\right) = Z^0_{sx}\left(\frac{\alpha_s}{N}\right) + \alpha_s Z^1_{sx}\left(\frac{\alpha_s}{N}\right) + O(\alpha_s^2), \tag{3.3}$$

with the constraint that $Z^0_{sx}(0) = 1$. Upon scheme change, the leading-order contributions in the expansion of the BFKL and GLAP kernel in powers of $\alpha_s$ remain unchanged, while higher-order terms are modified. It is interesting to observe that the lowest-order nontrivial small-$x$ scheme change $Z^0_{sx}\left(\frac{\alpha_s}{N}\right)$ (which affects the NLO BFKL kernel $\chi_1(N)$ in eq. (2.25)) amounts to a leading-order redefinition of the normalization of gluon, i.e. it starts at order $\alpha_s^0$ in the expansion of $Z$ in powers of $\alpha_s$ at fixed $\frac{\alpha_s}{N}$ [17].

A further related complication is due to the fact that the BFKL equation is, in its most general form, given for a $k_\perp$-dependent parton distribution, and its form eq. (2.5) is obtained from angular averaging of this $k_\perp$-dependent parton distribution. This angular averaging leads to an "unintegrated" parton distribution $\mathcal{G}(N, t)$, which is the derivative of the usual parton distribution $G(N, t)$: $\mathcal{G}(N, t) = \frac{d}{dt} G(N, t)$ i.e., in Mellin space,

$$\mathcal{G}(N, M) = M G(N, M). \tag{3.4}$$

Because of eq. (2.17), the evolution kernels $\chi(\hat{\alpha}_s, M)$ and $\chi_i(\hat{\alpha}_s, M)$ for the BFKL equations satisfied by respectively $\mathcal{G}(M, N)$ and $G(M, N)$ do not coincide, and are related by

$$\chi(\hat{\alpha}_s, M) = M \chi_i(\hat{\alpha}_s, M) M^{-1}. \tag{3.5}$$

Hence, if the BFKL kernel is determined for the unintegrated gluon distribution $\mathcal{G}(N,t)$, while the GLAP anomalous dimension refers to the standard integrated quantity $G(N,t)$, there are extra contributions to the duality relation, due to the non-commutativity of $M$ and $\hat{\alpha}_s$ in eq. (3.5). The relation between integrated and unintegrated parton distributions eq. (3.4) can be viewed as a scheme change by using the $(N,t)$ space form of the GLAP equation (2.4) to relate $\mathcal{G}$ to $G$, namely:

$$\mathcal{G}(N,t) = \gamma(\alpha_s(t), N) G(N,t). \tag{3.6}$$

Note that this is not a proper scheme change eq. (3.3) because it does not reduce to the identity in the limit $\alpha_s \to 0$. Note also that just as the evolution kernels for $\mathcal{G}$ and $G$ do not coincide, similarly the anomalous dimension for their evolution differ: in fact, from eq. (3.4) and the GLAP equation (2.4) one finds that $\mathcal{G}$ satisfies

$$\frac{d}{dt}\mathcal{G}(N,t) = \gamma_u(\alpha_s(t), N)\mathcal{G}(N,t), \tag{3.7}$$

where

$$\gamma_u(\alpha_s(t), N) = \gamma(\alpha_s(t), N) + \frac{d}{dt}\ln\left[\gamma(\alpha_s(t), N)\right]. \tag{3.8}$$

Henceforth, we shall denote by $\gamma$ the anomalous dimension at the integrated level, and by $\chi$ the kernel at the unintegrated level, and by $\gamma_u$ and $\chi_i$ the unintegrated anomalous dimension and integrated kernel.

Let us now come to the scheme choices which are relevant for the explicit determination of $\gamma(\alpha_s, N)$ and $\chi(\alpha_s, M)$. The normalization of the parton distribution which appears in the GLAP equation (2.4) is fixed by the standard factorization of collinear singularities [27], and a choice of subtraction prescription such as e.g. dimensional regularization and the $\overline{\rm MS}$ prescription. This defines anomalous dimensions in the $\overline{\rm MS}$ factorization scheme. Duality then implicitly defines a corresponding factorization scheme for the BFKL equation. However, the direct computation of the next-to-leading order BFKL kernel is based on the determination [12,13,28] of the gluon Green function in the high-energy limit. The extraction of the large energy behavior of the gluon Green function, and the resummation of its large-energy logs through the BFKL equation are based (explicitly [13] or implicitly [12,28]) on a factorization of cross-sections in terms of a high-energy parton distribution (the so-called $k_\perp$ factorization [29]) which is compatible with the usual collinear factorization, but differs from it by a computable scheme change.

This is due to the fact that the usual parton distribution which enters the collinear-factorized GLAP equation, and the gluon density which enters the $k_\perp$-factorization formula are normalized differently. This means that, even though the gluon Green-function itself is computed in the $\overline{\rm MS}$ scheme, the evolution kernel extracted from it corresponds to a scheme which is not $\overline{\rm MS}$, because it describes evolution of a quantity which differs from the $\overline{\rm MS}$ parton distribution by a normalization factor, i.e., it can be obtained from the $\overline{\rm MS}$ parton distribution by a suitable scheme-change function $Z_{sx}$ eq. (3.3). This scheme-change function defines the so-called $Q_0$ factorization scheme [16]. Furthermore, the quantity which naturally enters $k_\perp$-factorization formulae is the unintegrated parton distribution

$\mathcal{G}(N,t)$, so $Q_0$ scheme results are usually given for this quantity, though, of course, $Q_0$ scheme results can also be given for the integrated parton distribution by using eq. (3.5) and the corresponding relation eq. (3.7) between anomalous dimensions.

The normalization mismatch between $k_\perp$ factorization and collinear factorization, and thus the precise definition of the $Q_0$ scheme, has been determined in ref. [29] at leading nontrivial order, i.e. at the level of $Z^0_{sx}$ eq. (3.3), which affects the definition of $\chi_1$ in the expansion eq. (2.25) of the BFKL kernel $\chi$, and therefor its dual GLAP anomalous dimension $\gamma$ up to next-to-leading order (order $\gamma_{ss}$) in the expansion of $\gamma(\alpha_s, N)$ in powers of $\alpha_s$ at fixed $\frac{\alpha_s}{N}$:

$$\gamma(\alpha_s, N) = \gamma_s\left(\frac{\alpha_s}{N}\right) + \alpha_s\gamma_{ss}\left(\frac{\alpha_s}{N}\right) + \dots. \tag{3.9}$$

The scheme change at the next order (relevant for $\chi_2$ and $\gamma_{sss}$) has been recently derived in ref. [18].[1]

The main result of ref. [18], which we shall use in what follows, is an expression (proven up to NNLO, but conjectured to hold in general) which relates the $t$ dependence of the integrated parton distributions $G(N,t)$ (as defined in standard collinear factorization) in the $\overline{\rm MS}$ and $Q_0$ scheme. This relation is expressed in terms of the BFKL kernel for the unintegrated distribution $\mathcal{G}(N,t)$. Specifically, the $t$ dependence can be written in terms of a saddle-point evolution factor $E(t,t_0)$, a running–coupling duality correction $\mathcal{N}(N,t)$, and a normalization factor $\mathcal{R}(t_0)$ which is characteristic of the way minimal subtraction with dimensional regularization is defined by continuation of the anomalous dimensions in $d$ dimensions.

The saddle-point evolution factor is obtained by solving the running–coupling BFKL equation for the unintegrated distribution in the saddle-point approximation: this can be shown [30,26,23] to lead to evolution driven by the anomalous dimension $\tilde\gamma_u(\alpha_s(t), N)$, obtained from the unintegrated BFKL kernel using naive (fixed-coupling) duality eq. (2.7), but with $\alpha_s = \alpha_s(t)$,

$$E(t,t_0) = \exp\left[\int_{t_0}^{t} \tilde\gamma_u(\alpha_s(t'), N)\, dt'\right]. \tag{3.10}$$

The running–coupling correction to duality discussed in the previous section can be combined with the factor eq. (3.6) which relates the integrated and unintegrated parton distributions. This gives

$$\begin{aligned}\frac{\mathcal{N}(N,t)}{\mathcal{N}(N,t_0)} &= \frac{\gamma(\alpha_s(t_0),N)}{\gamma(\alpha_s(t),N)} \exp\left\{\int_{t_0}^{t}[\gamma_u(\alpha_s(t'),N) - \tilde\gamma_u(\alpha_s(t'),N)]\,dt'\right\}\\ &= \exp\left\{\int_{t_0}^{t}[\gamma(\alpha_s(t'),N) - \tilde\gamma_u(\alpha_s(t'),N)]\,dt'\right\}.\end{aligned} \tag{3.11}$$

[1] Note that in ref. [18] (and elsewhere) the scheme change effected by $Z^0_{sx}$ is referred to as a leading-log $x$ scheme change, essentially because it is a leading-order redefinition of the gluon normalization, whereas we will call it a NLO scheme change, because it affects the NLO kernel $\chi_1$.

Note that $\lim_{\alpha_s \to 0} \mathcal{N}(N,t) = 1$ so this can be viewed as a scheme change in the proper sense.

Finally, the normalization factor $\mathcal{R}$ is due to the fact that in the $\overline{\rm MS}$ scheme the anomalous dimension is defined as the residue of the simple $\varepsilon$ pole in the partonic cross section, which in turn is given by

$$\begin{aligned}\frac{\gamma(\alpha_s,N,\varepsilon)}{\beta(\alpha_s,\varepsilon)} &= \frac{1}{\alpha_s\varepsilon}\left(1-\frac{\beta(\alpha_s)}{\alpha_s\varepsilon}+\ldots\right)(\gamma(\alpha_s,N)+\varepsilon\dot\gamma(\alpha_s,N)+\ldots)\\ &= \frac{1}{\alpha_s\varepsilon}\left(\gamma(\alpha_s,N)-\frac{\beta(\alpha_s)}{\alpha_s}\dot\gamma(\alpha_s,N)+\ldots\right),\end{aligned} \tag{3.12}$$

where in $\overline{\rm MS}$ $\beta(\alpha_s,\varepsilon)=\alpha_s\varepsilon+\beta(\alpha_s)$ is the $d$-dimensional $\beta$ function, $\gamma(\alpha_s,\varepsilon)$ is the anomalous dimension obtained using duality from a $d$-dimensional BFKL kernel, which has been expanded as

$$\gamma(\alpha_s,N,\varepsilon)=\gamma(\alpha_s,N)+\varepsilon\dot\gamma(\alpha_s,N)+O(\varepsilon^2). \tag{3.13}$$

It follows that the anomalous dimension determined from duality differs from the $\overline{\rm MS}$ result through the terms beyond the first on the r.h.s. of eq. (3.12), and thus $\overline{\rm MS}$ evolution requires an additional factor

$$\frac{\mathcal{R}(N,t)}{\mathcal{R}(N,t_0)} = \exp\left[-\int_{t_0}^{t}\beta_0\alpha(t')\dot\gamma(\alpha(t'),N)dt'+O(\alpha_s^2)\right], \tag{3.14}$$

where we have used $\beta(\alpha_s)=-\beta_0\alpha_s^2+\ldots$.

Combining all these factors, the result of ref. [18] takes the form:

$$G^{Q_0}(N,t)=\mathcal{N}(N,t)E(t,t_0)\mathcal{R}(N,t_0)G^{\overline{\rm MS}}(N,t_0) \tag{3.15}$$

This equation gives the scale dependence of the parton distribution in either scheme in terms of a boundary condition determined in the other scheme: therefore, it fully specifies both the relation between the two schemes, and the scale dependence in either of them. Letting $t=t_0$ in eq. (3.15) immediately gives the relation between $G^{Q_0}(N,t)$ and $G^{\overline{\rm MS}}(N,t)$, through the function

$$R(N,t)\equiv\mathcal{N}(N,t)\mathcal{R}(N,t). \tag{3.16}$$

The scale dependence of the parton distribution in the $Q_0$ scheme is found by keeping $t_0$ fixed in eq. (3.15) and varying $t$: thus

$$\begin{aligned}G^{Q_0}(N,t) &= \frac{\mathcal{N}(N,t)}{\mathcal{N}(N,t_0)}E(t,t_0)G^{Q_0}(N,t_0)\\ &= \exp\left[\int_{t_0}^{t}\gamma(\alpha_s(t'),N)\,dt'\right]G^{Q_0}(N,t_0).\end{aligned} \tag{3.17}$$

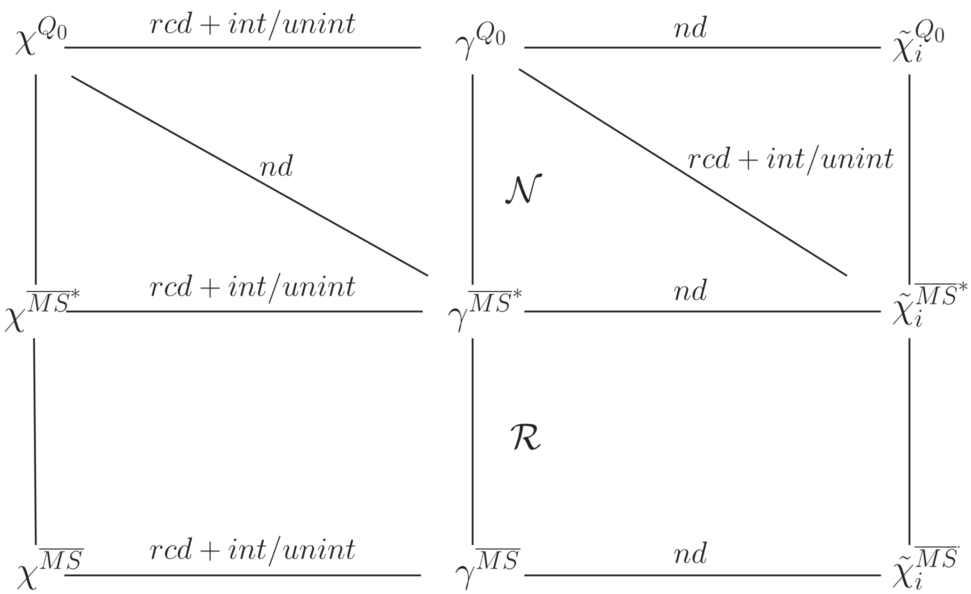


Figure 2: Schematic relation between BFKL kernels $\chi$ and GLAP anomalous dimensions $\gamma$ in various schemes. Horizontal lines denote duality relations while vertical lines denote scheme transformations. The diagonal lines express the identities eq. (3.21) and (3.22).

So the integrated parton distribution in the $Q_0$ scheme evolves with the anomalous dimension $\gamma(N, t)$ which is related to the starting unintegrated BFKL kernel by running–coupling duality combined with the transformation to the integrated level.

The scale dependence in the $\overline{\rm MS}$ scheme is instead given by

$$\begin{aligned} G^{\overline{\rm MS}}(N,t) &= \frac{\mathcal{R}(N,t_0)}{\mathcal{R}(N,t)} E(t,t_0) G^{\overline{\rm MS}}(N,t_0) \\ &= \frac{\mathcal{R}(N,t_0)}{\mathcal{R}(N,t)} \exp\left[\int_{t_0}^{t} \tilde{\gamma}_u(\alpha_s(t'),N)\,dt'\right] G^{\overline{\rm MS}}(N,t_0), \end{aligned} \tag{3.18}$$

namely, the integrated parton distribution in the $\overline{\rm MS}$ scheme evolves with an anomalous dimension which is closely related to the fixed-coupling dual $\tilde{\gamma}_u(N, t)$ of the starting (unintegrated) BFKL kernel, and only differs from it through the scale dependence of the $\mathcal{R}$ factor eq. (3.14). In this respect the $\overline{\rm MS}$ scheme is remarkably simple.

In fact it is possible, and useful, to define an auxiliary scheme, $\overline{\rm MS}^*$, which differs from $\overline{\rm MS}$ by a factor $\mathcal{R}(N, t)$: namely

$$G^{\overline{\rm MS}^*}(N,t) = \mathcal{R}(N,t) G^{\overline{\rm MS}}(N,t) \tag{3.19}$$

In this $\overline{\rm MS}^*$ scheme the parton distribution evolves with the naive dual anomalous dimension:

$$G^{\overline{\rm MS}^*}(N,t_1) = \exp\left[\int_{t_0}^{t_1} \tilde{\gamma}_u(\alpha_s(t),N)\,dt\right] G^{\overline{\rm MS}^*}(N,t_0). \tag{3.20}$$

The relations between different quantities in various schemes, which will be computed in the next section, are summarized in figure 2. In the figure, horizontal lines denote duality: either at the running–coupling level, relating $\chi$ to $\gamma$, or at the fixed-coupling, relating $\gamma$ to $\tilde{\chi}$. Vertical lines denote relations between schemes, specifically the $\mathrm{Q}_0$, $\overline{\mathrm{MS}}^*$ and $\overline{\mathrm{MS}}$ schemes. Equation (3.20) means that the anomalous dimension in the $\overline{\mathrm{MS}}^*$ scheme coincides with the naive dual of the $\mathrm{Q}_0$ scheme BFKL kernel (at the unintegrated level):

$$\tilde{\gamma}_u^{\mathrm{Q}_0}(\alpha_s, N) = \gamma^{\overline{\mathrm{MS}}^*}(\alpha_s, N), \tag{3.21}$$

and thus, by duality, also that

$$\tilde{\chi}_i^{\overline{\mathrm{MS}}^*}(\alpha_s, M) = \chi^{Q0}(\alpha_s, M), \tag{3.22}$$

where $\tilde{\chi}_i^{\overline{\mathrm{MS}}^*}(\alpha_s, M)$ is the naive dual of the standard GLAP anomalous dimension in the $\overline{\mathrm{MS}}^*$ scheme, while $\chi^{Q0}(\alpha_s, M)$ is the kernel for the BFKL equation satisfied by the unintegrated parton distribution in the $\mathrm{Q}_0$ scheme. This further implies that if we interpret $\tilde{\chi}_i^{\overline{\mathrm{MS}}^*}(\alpha_s, M)$ as an operator by letting $\alpha_s \to \hat{\alpha}_s$, and order it canonically (i.e. in the same way as $\chi^{Q0}(\hat{\alpha}_s, M)$), then it is related by running–coupling duality to $\gamma^{\mathrm{Q}_0}(\alpha_s, N)$. These relations are denoted by diagonal lines in the figure, and will turn out to be useful in relating results obtained in various schemes in the next section.

## 4. The NNLO BFKL kernel

The construction of the NNLO BFKL kernel proceeds in three steps. First, we collect the results discussed in the previous sections to get expressions of the BFKL kernels which are related by either running–coupling duality ($\chi(\hat{\alpha}_s, M)$) or fixed-coupling duality ($\tilde{\chi}(\alpha_s, M)$) to a given GLAP kernel $\gamma(\alpha_s, N)$ to NNLO in various schemes. Then, we use the known expression of the GLAP anomalous dimensions up to next-to-next-to leading order to determine the first three coefficients of the expansion eq. (2.26) of the leading $\chi_0(\hat{\alpha}_s, M)$, next-to-leading $\chi_1(\hat{\alpha}_s, M)$, and next-to-next-to-leading $\chi_2(\hat{\alpha}_s, M)$ order BFKL kernels in powers of $M$ about $M = 0$. Finally, we use the underlying symmetry of the high-energy gluon emission diagrams to determine the corresponding coefficients of the expansion of $\chi_0(\hat{\alpha}_s, M)$, $\chi_1(\hat{\alpha}_s, M)$ and $\chi_2(\hat{\alpha}_s, M)$, about $M = 1$.

Because all duality relations are expressed in terms of naive (fixed-coupling) dual kernels, our starting point is the determination of the BFKL kernel $\tilde{\chi}(\alpha_s, M)$ which is obtained from the NNLO GLAP anomalous dimension using the fixed-coupling duality equation (2.8). Expanding $\tilde{\chi}(\alpha_s, M)$ in powers of $\alpha_s$ at fixed $\alpha_s/M$ as

$$\tilde{\chi}(\alpha_s, M) = \tilde{\chi}_s(\alpha_s/M) + \alpha_s \tilde{\chi}_{ss}(\alpha_s/M) + ... \tag{4.1}$$

we have

$$\gamma_0\big(\tilde{\chi}_s(\alpha_s/M)\big) = \frac{M}{\alpha_s}, \tag{4.2}$$

$$\tilde{\chi}_{ss}(\alpha_s/M) = -\frac{\gamma_1\big(\chi_s(\alpha_s/M)\big)}{\gamma_0'\big(\chi_s(\alpha_s/M)\big)}, \tag{4.3}$$

$$\begin{aligned}\tilde{\chi}_{sss}(\alpha_s/M) = -\frac{1}{\gamma_0'\big(\chi_s(\alpha_s/M)\big)}\Bigg[&\gamma_2\big(\chi_s(\alpha_s/M)\big) + \gamma_1'\big(\chi_s(\alpha_s/M)\big)\chi_{ss}(\alpha_s/M) \\ &+ \frac{1}{2}\gamma_0''\big(\chi_s(\alpha_s/M)\big)\big(\chi_{ss}(\alpha_s/M)\big)^2\Bigg].\end{aligned} \tag{4.4}$$

Expanding out both $\gamma_i$ and $\tilde{\chi}_{s^i}$, $i = 1, 2, 3$ up to third order in their respective arguments, eqs. (4.2)-(4.4) determine the coefficients of these expansions in terms of each other. Starting with the expansion of $\gamma$

$$\gamma_0(N) = \frac{g_{0,-1}}{N} + g_{0,0} + g_{0,1}N + O(N^2), \tag{4.5}$$

$$\gamma_1(N) = \frac{g_{1,-1}}{N} + g_{1,0} + O(N), \tag{4.6}$$

$$\gamma_2(N) = \frac{g_{2,-2}}{N^2} + \frac{g_{2,-1}}{N} + O(N^0), \tag{4.7}$$

we get

$$\tilde{\chi}_0^i(M) = \frac{g_{0,-1}}{M} + O(M^2), \tag{4.8}$$

$$\tilde{\chi}_1^i(M) = \frac{g_{0,-1}g_{0,0}}{M^2} + \frac{g_{1,-1}}{M} + \frac{g_{2,-2}}{g_{0-1}} + O(M), \tag{4.9}$$

$$\tilde{\chi}_2^i(M) = \frac{(g_{0,-1})^2 g_{0,1} + g_{0,-1}(g_{0,0})^2}{M^3} + \frac{g_{1,0}g_{0,-1} + g_{0,0}g_{1,-1}}{M^2} + \frac{g_{2,-1}}{M} + O(M^0), \tag{4.10}$$

where the superscript $i$ reminds us that all these quantities are defined at the integrated level. The values of the coefficients $g_{i,j}$ up to NNLO in the $\overline{\text{MS}}$ scheme can be extracted from the known GLAP anomalous dimensions [31], and are listed in appendix B. Using these expressions of $g_{i,j}$, eqs. (4.8)-(4.10) give the first three terms in the expansion of the naive dual kernel $\tilde{\chi}$ in the same scheme: this corresponds to relation between $\gamma^{\overline{\text{MS}}}$ and $\tilde{\chi}^{\overline{\text{MS}}}$ in figure 2. Note that the vanishing of leading singularities of $\gamma$ at NLO and NNLO, $g_{1,-2} = g_{2,-3} = 0$, implies the well-known vanishing of the constant and linear term in the LO BFKL kernel (4.8).

Starting from the naive dual kernel $\tilde{\chi}_k^i$ eq. (4.8)-(4.10) we can determine the BFKL kernel in the same approximation in various factorization schemes. For the sake of comparison with direct diagrammatic computations, the $\text{Q}_0$ scheme is most relevant, because a direct computation using minimal subtraction at the level of the BFKL equation gives the kernel in this scheme: indeed, available expressions [12,13,28] of the next-to-leading kernel $\chi_1$ provide the result in the $\text{Q}_0$ scheme. Using duality, this kernel can be obtained from the $\overline{\text{MS}}$ GLAP anomalous dimension by exploiting eq. (3.21) (see figure 2), which states that $\chi^{\text{Q}_0}$ is the naive dual of the (integrated) anomalous dimension $\gamma^{\overline{\text{MS}}^*}$. Hence, we can obtain $\chi^{\text{Q}_0}$ by first transforming the anomalous dimension to the $\overline{\text{MS}}^*$ scheme through the $\mathcal{R}$ scheme change eq. (3.14) and then using naive duality eqs. (4.8)-(4.10).

The scheme change which takes the anomalous dimension to $\overline{\rm MS}^*$ can be determined, using eq. (3.14), from the expression of the GLAP splitting functions in $d = 4 - 2\varepsilon$ dimensions $\gamma(N, \varepsilon)$, as explained in Appendix C. The $\overline{\rm MS}^*$ anomalous dimension is given by

$$\gamma_0^{\overline{\rm MS}^*}(N) = \frac{g_{0,-1}}{N} + g_{0,0} + g_{0,1}N + O(N^2), \tag{4.11}$$

$$\gamma_1^{\overline{\rm MS}^*}(N) = \frac{g_{1,-1}}{N} + \bar{g}_{1,0} + O(N), \tag{4.12}$$

$$\gamma_2^{\overline{\rm MS}^*}(N) = \frac{g_{2,-2}}{N^2} + \frac{\bar{g}_{2,-1}}{N} + O(N^0), \tag{4.13}$$

where all coefficients are the same as in the $\overline{\rm MS}$ eqs. (4.5)-(4.7) except

$$\bar{g}_{1,0} = g_{1,0} - \beta_0 \dot{g}_{0,0}\,, \qquad \bar{g}_{2,-1} = g_{2,-1} - \beta_0 \dot{g}_{1,-1} - \beta_0^2 \ddot{g}_{0,-1}. \tag{4.14}$$

The coefficients $\dot{g}_{0,0}$ and $\ddot{g}_{0,-1}$ are determined by the $d$-dimensional LO splitting functions; their values are given in Appendix C. The coefficient $\dot{g}_{1,-1}$ is determined by the $d$–dimensional NLO GLAP kernel which is as yet unknown. This is the only term contributing to the singular part of the NNLO BFKL kernel that we have not been able to calculate explicitly: we will estimate below the uncertainty related to our ignorance of this contribution to the scheme change.

The BFKL kernel in the $\rm Q_0$ scheme is thus given in terms of the naive dual kernel eq. (4.8)-(4.10) by

$$\chi_0^{\rm Q_0}(M) = \tilde{\chi}_0^i(M), \tag{4.15}$$

$$\chi_1^{\rm Q_0}(M) = \tilde{\chi}_1^i(M) + O(M), \tag{4.16}$$

$$\chi_2^{\rm Q_0}(M) = \frac{(g_{0,-1})^2 g_{0,1} + g_{0,-1}(g_{0,0})^2}{M^3} + \frac{\bar{g}_{1,0}g_{0,-1} + g_{0,0}g_{1,-1}}{M^2} + \frac{\bar{g}_{2,-1}}{M} + O(M^0) \tag{4.17}$$

We can also determine the BFKL kernel in the $\overline{\rm MS}$ scheme, by observing that it is related by running–coupling duality to the $\overline{\rm MS}$ anomalous dimension; note that this now gives the kernel at the integrated level (figure 2 again). Hence, we add to eqs. (4.8)-(4.10) the running–coupling corrections discussed in Sect. 2, and explicitly given in appendix A, eqs. (A.13)-(A.15) in terms of the naive dual expressions computed above. Defining

$$\Delta^{\overline{\rm MS}}\chi(M) = \chi_k^{i,\overline{\rm MS}}(M) - \tilde{\chi}_k^i(M), \tag{4.18}$$

and expanding about $M = 0$ we get

$$\Delta^{\overline{\rm MS}}\chi_0(M) = 0 \tag{4.19}$$

$$\Delta^{\overline{\rm MS}}\chi_1(M) = \beta_0 \frac{g_{0,-1}}{M^2} \tag{4.20}$$

$$\Delta^{\overline{\rm MS}}\chi_2(M) = \beta_0 \left( \frac{3g_{0,0}g_{0,-1}}{M^3} + \frac{2g_{1,-1}}{M^2} + \frac{2g_{2,-2}}{g_{0,-1}M} \right) + 2\beta_0^2 \frac{g_{0,-1}}{M^3} + \beta_0\beta_1 \frac{g_{0,-1}}{M^2}. \tag{4.21}$$

Using eqs.(4.19)-(4.21) in eq. (4.18) we get the BFKL kernel in the $\overline{\rm MS}$ scheme at the integrated level. The expression at the unintegrated level, to be compared to the unintegrated $Q_0$ scheme result of eqs. (4.15)-(4.17) can be obtained using using eq. (3.4), which implies

$$\frac{d}{d\xi}\mathcal{G}(\xi,M) = \Big(\hat\alpha_s\chi_0 + \hat\alpha_s^2\chi_1 + \hat\alpha_s^3\chi_2 - [\hat\alpha_s, M]\,\frac{\chi_0}{M} - \big[\hat\alpha_s^2, M\big]\,\frac{\chi_1}{M} + O(\hat\alpha_s^4)\Big)\,\mathcal{G}(\xi,M), \tag{4.22}$$

so the kernels $\chi_i^i$ at the integrated level and $\chi_i$ at the unintegrated level are related by

$$\begin{aligned}
\chi_0 &= \chi_0^i,\\
\chi_1 &= \chi_1^i - \beta_0\frac{\chi_0^i}{M},\\
\chi_2 &= \chi_2^i - \beta_0\beta_1\frac{\chi_0^i}{M} - 2\beta_0\frac{\chi_1^i}{M}.
\end{aligned} \tag{4.23}$$

So far, we have exploited the information from the GLAP anomalous dimension to determine the first few terms in the expansion of the BFKL kernel about $M=0$ (the collinear region). However, we can further use the underlying symmetry of the BFKL kernel to determine the corresponding terms in the expansion of the BFKL kernel about $M=1$ (the anticollinear region). At LO and NLO this leads to an approximation to the BFKL kernel which, when tested on the known LO and NLO kernels, turns out to be accurate to better than 2% in the whole physical region $0<M<1$, as shown in figure 1. Indeed, the underlying Feynman diagrams which determine the unintegrated BFKL kernel are symmetric upon the exchange of the incoming and outgoing gluon. This means that the dimensionless BFKL kernel $K(\alpha_s,k^2,Q^2)$, related to $\chi(\hat\alpha_s,M)$ by Mellin transformation

$$\chi(\alpha_s,M) = \int_0^\infty \frac{dQ^2}{Q^2}\left(\frac{Q^2}{k^2}\right)^{-M} K\left(\alpha_s,k^2,Q^2\right), \tag{4.24}$$

is symmetric upon the interchange of the virtualities of the incoming gluons $Q^2$ and $k^2$:

$$\frac{1}{Q^2}K(\alpha_s,k^2,Q^2) = \frac{1}{k^2}K(\alpha_s,Q^2,k^2). \tag{4.25}$$

This symmetry, in turn, implies that the BFKL kernel $\chi(\alpha_s,M)$ is symmetric upon the interchange $M\leftrightarrow 1-M$ [12].

However, the kernel is symmetric only if one chooses a symmetric argument for the running coupling, and if the $N$-Mellin eq. (2.2) does not break the symmetry between the two scales $Q^2$ and $\mu^2$ which enter the definition eq. (2.3) of the $M$-Mellin transform. In deep–inelastic scattering both symmetries are broken: the argument of the running coupling is $Q^2$, and $\xi=\ln(s/Q^2)$. Hence, the BFKL kernel which is obtained from the GLAP anomalous dimensions incorporates these symmetry-breaking effects. The symmetry breaking must be undone before the symmetry can be exploited. After symmetrizing one can revert to nonsymmetric variables and argument of the coupling in order to get an expression of the dual $\chi$ which is accurate for all $M$.

A symmetric choice of variables is for example $\xi = \ln(s/\sqrt{Q^2k^2})$. As is well known [12], the kernel $\chi^s(\hat{\alpha}_s, M)$ which corresponds to this choice is related to the kernel $\chi^{\rm DIS}(\hat{\alpha}_s, M)$ which corresponds to DIS variables by the implicit equation

$$\chi^s(\hat{\alpha}_s, M) = \chi^{\rm DIS}\left(\hat{\alpha}_s, M + \tfrac{1}{2}\chi^s(\hat{\alpha}_s, M)\right). \tag{4.26}$$

A symmetric choice of argument for the running coupling is such that eq. (4.25) also holds when the argument $\mu^2$ of the running coupling $\alpha_s(\mu^2)$ is expressed as a function of $Q^2$ and $k^2$, $\mu^2 = \mu^2(Q^2, k^2)$, and $\mu^2(Q^2, k^2) = \mu^2(k^2, Q^2)$. Examples of symmetric choices for the running of the coupling are $\mu^2 = |Q^2 - k^2|$ or $\mu^2 = {\rm Max}(Q^2, k^2)$. Upon Mellin transformation, different choices of arguments of the running coupling correspond to different orderings of the running–coupling operator. For instance, it is easy to check that the Mellin transform of

$$K_0(\alpha_s, k^2, Q^2) = \frac{C_A}{\pi}\alpha_s(Q^2)\frac{Q^2}{|Q^2 - k^2|} \tag{4.27}$$

is $\hat{\alpha}_s\chi_0(M)$, where

$$\chi_0(M) = -\frac{C_A}{\pi}\left[\psi(M) + \psi(1-M) - 2\psi(1)\right] \tag{4.28}$$

is the standard leading-order BFKL kernel, while the Mellin transform of

$$K_0(\alpha_s, k^2, Q^2) = \frac{C_A}{\pi}\alpha_s(k^2)\frac{Q^2}{|Q^2 - k^2|} \tag{4.29}$$

is $\chi_0(M)\hat{\alpha}_s$.

Note finally that the integration which takes us from the unintegrated to the integrated distribution also breaks the symmetry, as eq. (3.5) explicitly shows: it follows that the symmetry which holds at the unintegrated level is broken at the integrated level.

Because the diagrammatic computation of the BFKL kernel yields the result in the $Q_0$ scheme at the unintegrated level, once symmetric variables and symmetric running of the coupling have been chosen the kernel is symmetric when determined in this scheme. Whether the symmetry is preserved or not in other schemes depends of course on the particular scheme change. Specifically, it is easy to see that a scheme change through $\mathcal{R}(N,t)$ preserves the symmetry at leading nontrivial order (i.e. at the order of $\chi_1$) but not beyond, essentially because it is entirely determined by $\chi_0$ and its $d$-dimensional continuation, which is symmetric (see appendix C), whereas a scheme change through $\mathcal{N}(N,t)$ breaks the symmetry already at leading nontrivial order, because the first nontrivial running–coupling duality correction eq. (2.23) is manifestly not symmetric.

As discussed already, use of naive duality on the $\overline{\rm MS}^*$ GLAP anomalous dimension eqs. (4.11)-(4.13) gives us the expansion eqs. (4.15)-(4.17) of the $Q_0$ scheme BFKL kernel for the unintegrated parton distribution. This result is clearly in DIS variables and can be

turned into symmetric variables by expanding out eq. (4.26). Substituting the expansion eq. (2.25) of the kernel we get

$$\begin{aligned}\chi^s(\hat\alpha_s, M) &= \hat\alpha_s\chi_0\left(M + \tfrac{1}{2}\hat\alpha_s\chi_0^s(M) + \tfrac{1}{2}\hat\alpha_s^2\chi_1^s(M)\right)\\ &\qquad + \hat\alpha_s^2\chi_1\left(M + \tfrac{1}{2}\hat\alpha_s\chi_0^s(M)\right) + \hat\alpha_s^3\chi_2(M) + O(\hat\alpha_s^4) \qquad (4.30)\\ &= \hat\alpha_s\chi_0\left(M + \tfrac{1}{2}\hat\alpha_s\chi_0^s(M)\right) + \hat\alpha_s^3\chi_0'\left(M + \tfrac{1}{2}\hat\alpha_s\chi_0^s(M)\right)\tfrac{1}{2}\chi_1^s(M)\\ &\qquad + \hat\alpha_s^2\chi_1(M) + \hat\alpha_s^3\chi_1'(M)\tfrac{1}{2}\chi_0^s(M) + \hat\alpha_s^3\chi_2(M) + O(\hat\alpha_s^4). \qquad (4.31)\end{aligned}$$

where on the right-hand side we have dropped the DIS index of eq. (4.26).

The first term on the right-hand side of eq. (4.31) must be computed by carefully keeping operator ordering into account. This can be done by using a technique akin to that of ref. [7], summarized in section 2: namely, by computing

$$\begin{aligned}\chi_0(M + \tfrac{1}{2}\hat\alpha_s\chi_0^s(M)) &= e^{\frac{1}{2}\hat\alpha_s\chi_0^s(M)\frac{d}{d\lambda} + \frac{1}{2}\left[M, \frac{1}{2}\hat\alpha_s\chi_0^s(M)\right]\frac{d^2}{d\lambda^2} + \dots}\chi_0(M+\lambda)|_{\lambda=0}\\ &= \chi_0(M) + \tfrac{1}{2}\hat\alpha_s\chi_0^s(M)\chi_0'(M) + \tfrac{1}{4}\left[M, \hat\alpha_s\chi_0^s(M)\right]\chi_0''(M) + \tfrac{1}{8}\hat\alpha_s^2{\chi_0^s}^2(M)\chi_0''(M) + O(\hat\alpha_s^3)\\ &= \chi_0(M) + \tfrac{1}{2}\hat\alpha_s\chi_0^s(M)\chi_0'(M) - \tfrac{1}{4}\beta_0\hat\alpha_s^2\chi_0^s(M)\chi_0''(M) + \tfrac{1}{8}\hat\alpha_s^2{\chi_0^s}^2(M)\chi_0''(M) + O(\hat\alpha_s^3).\end{aligned} \qquad (4.32)$$

Substituting this result in eq. (4.31) and collecting terms of the same order we get

$$\chi_0^s(M) = \chi_0^{\mathrm{Q}_0}(M), \qquad (4.33)$$

$$\chi_1^s(M) = \chi_1^{\mathrm{Q}_0}(M) + \tfrac{1}{2}{\chi_0'}^{\mathrm{Q}_0}(M)\chi_0^{\mathrm{Q}_0}(M), \qquad (4.34)$$

$$\begin{aligned}\chi_2^s(M) &= \chi_2^{\mathrm{Q}_0}(M) + \tfrac{1}{2}{\chi_1'}^{\mathrm{Q}_0}(M)\chi_0^{\mathrm{Q}_0}(M) + \tfrac{1}{2}{\chi_0'}^{\mathrm{Q}_0}(M)\chi_1^{\mathrm{Q}_0}(M)\\ &\quad + \tfrac{1}{2}{{\chi_0'}^{\mathrm{Q}_0}(M)}^2\chi_0^{\mathrm{Q}_0}(M) + \tfrac{1}{8}{\chi_0''}^{\mathrm{Q}_0}(M){\chi_0^{\mathrm{Q}_0}}^2(M) - \tfrac{1}{4}\beta_0{\chi_0''}^{\mathrm{Q}_0}(M)\chi_0^{\mathrm{Q}_0}(M), \qquad (4.35)\end{aligned}$$

where on the right-hand side we have denotes the result in DIS variables by $\chi_k^{\mathrm{Q}_0}$, since in practice we will use the $\mathrm{Q}_0$-scheme result in DIS variables of eqs. (4.15)-(4.17).

In order to determine the constant term of $\chi_1$ and the simple pole of $\chi_2$ in symmetric variables we have substituted the expansion of $\chi_0$ up to $O(M^2)$. In principle the linear term of $\chi_1$ is needed too, but its dependence cancels out in the expression for $\chi_2$. Using the expressions eq. (4.15)-(4.17) for the unintegrated $\mathrm{Q}_0$–scheme kernels $\chi_i^{\mathrm{Q}_0}$ we finally get

$$\chi_0^s(M) = \frac{g_{0,-1}}{M} + O(M^2)\,, \qquad (4.36)$$

$$\chi_1^s(M) = -\frac{(g_{0,-1})^2}{2M^3} + \frac{g_{0,0}g_{0,-1}}{M^2} + \frac{g_{1,-1}}{M} + \frac{g_{2,-2}}{g_{0,-1}} + g_{0,-1}^2\zeta(3) + O(M)\,, \qquad (4.37)$$

$$\begin{aligned}\chi_2^s(M) &= \frac{(g_{0,-1})^3}{2M^5} - \frac{3g_{0,0}(g_{0,-1})^2 + \beta_0(g_{0,-1})^2}{2M^4} + \frac{(g_{0,0})^2g_{0,-1} + g_{0,1}(g_{0,-1})^2 - g_{0,-1}g_{1,-1}}{M^3}\\ &\quad + \frac{-\frac{1}{2}g_{2,-2} + g_{0,0}g_{1,-1} + g_{0,-1}\bar g_{1,0}}{M^2} + \frac{\bar g_{2,-1} - 2\beta_0(g_{0,-1})^2\zeta(3)}{M} + O(M^0)\,. \qquad (4.38)\end{aligned}$$

We can now exploit the symmetry of the kernel with symmetric coupling, which implies that $\chi(\hat{\alpha}_s, M)$ must admit an expansion of the form

$$\chi(\hat{\alpha}_s, M) = \hat{\alpha}_s \chi_0^s(M) + \hat{\alpha}_s^2 \chi_1^s(M) + \hat{\alpha}_s^3 \chi_2^s(M) + O(\hat{\alpha}_s^3) \tag{4.39}$$
$$= \chi_0^{sym}(\hat{\alpha}_s, M) + \chi_1^{sym}(\hat{\alpha}_s, M) + \chi_2^{sym}(\hat{\alpha}_s, M) + O(\hat{\alpha}_s^3) \tag{4.40}$$

where $\chi_i^{sym}(\hat{\alpha}_s, M)$ are the symmetrized functions

$$\chi_0^{sym}(\hat{\alpha}_s, M) = c_{0,-1}\left[\hat{\alpha}_s \frac{1}{M} + \frac{1}{1-M}\hat{\alpha}_s\right] + \hat{\alpha}_s c_{0,0} + c_{0,1}\left[\hat{\alpha}_s M + (1-M)\hat{\alpha}_s\right]$$
$$+ c_{0,2}\left[\hat{\alpha}_s M^2 + (1-M)^2\hat{\alpha}_s\right] + O(M^3), \tag{4.41}$$
$$\chi_1^{sym}(\hat{\alpha}_s, M) = c_{1,-3}\left[\hat{\alpha}_s^2 \frac{1}{M^3} + \frac{1}{(1-M)^3}\hat{\alpha}_s^2\right] + c_{1,-2}\left[\hat{\alpha}_s^2 \frac{1}{M^2} + \frac{1}{(1-M)^2}\hat{\alpha}_s^2\right]$$
$$+ c_{1,-1}\left[\hat{\alpha}_s^2 \frac{1}{M} + \frac{1}{1-M}\hat{\alpha}_s^2\right] + \hat{\alpha}_s c_{1,0} + c_{1,1}\left[\hat{\alpha}_s^2 M + (1-M)\hat{\alpha}_s^2\right]$$
$$+ O(M^2), \tag{4.42}$$
$$\chi_2^{sym}(\hat{\alpha}_s, M) = \sum_{j=1,5} c_{2,-j}\left[\hat{\alpha}_s^3 \frac{1}{M^j} + \frac{1}{(1-M)^j}\hat{\alpha}_s^3\right] + O(M^0). \tag{4.43}$$

Given the expression of $\chi_i^{sym}$ eq. (4.40) it is straightforward to determine the symmetrized kernel when $\hat{\alpha}_s$ is "canonically" ordered to the left, which corresponds to the choice of argument of the running coupling $\alpha_s = \alpha_s(Q^2)$:

$$\hat{\alpha}_s \bar{\chi}_0^{sym}(M) = \chi_0^{sym}(\hat{\alpha}_s, M), \tag{4.44}$$
$$\hat{\alpha}_s^2 \bar{\chi}_1^{sym}(M) = \chi_1^{sym}(\hat{\alpha}_s, M) - \hat{\alpha}_s^2 \beta_0 \frac{c_{0,-1}}{(1-M)^2} + \beta_0 \hat{\alpha}_s^2 (c_{0,1} + 2c_{0,2}) + O(M), \tag{4.45}$$
$$\hat{\alpha}_s^3 \bar{\chi}_2^{sym}(M) = \chi_2^{sym}(\hat{\alpha}_s, M) - \hat{\alpha}_s^3 \beta_0 \beta_1 \frac{c_{0,-1}}{(1-M)^2} + 2\hat{\alpha}_s^3 \beta_0^2 \frac{c_{0,-1}}{(1-M)^3}$$
$$- 2\hat{\alpha}_s^3 \beta_0 \frac{c_{1,-1}}{(1-M)^2} - 4\hat{\alpha}_s^3 \beta_0 \frac{c_{1-2}}{(1-M)^3} - 8\hat{\alpha}_s^3 \beta_0 \frac{c_{1-3}}{(1-M)^4} + O(M^0). \tag{4.46}$$

In $\bar{\chi}_i^{sym}$ the symmetry is broken by the running of the coupling only.

We can finally determine all coefficients $c_{ij}$ in eqs. (4.41)-(4.43) by expanding the symmetrized kernel eqs. (4.44)-(4.46) in Laurent series about $M = 0$ and equating to the expansion of the unsymmetrized kernels $\chi_i^s$ eqs. (4.36)-(4.38), which is accurate to the stated power of $M$. Because the anticollinear terms with poles at $M = 1$ in eqs. (4.41)-(4.43) are regular in $M = 0$, the symmetrized $\chi_i^{sym}$ have the same $M = 0$ poles as their unsymmetrized counterparts $\chi_i^s$, and their coefficients can be read off eq. (4.36)-(4.38). However, the anticollinear terms do contribute to all regular contributions in the expansion of $\chi_i^s$ about $M = 0$. This is why higher–order regular terms must be included in the right-hand side of eqs. (4.41),(4.42): specifically, symmetric terms up to $O(M^2)$ must be included in $\chi_0^s(M)$ in order for its expansion to coincide with that of $\chi_0^{sym}(M)$ up to and including $O(M)$, and terms up to and including $O(M)$ in $\chi_1^s(M)$ in order for its

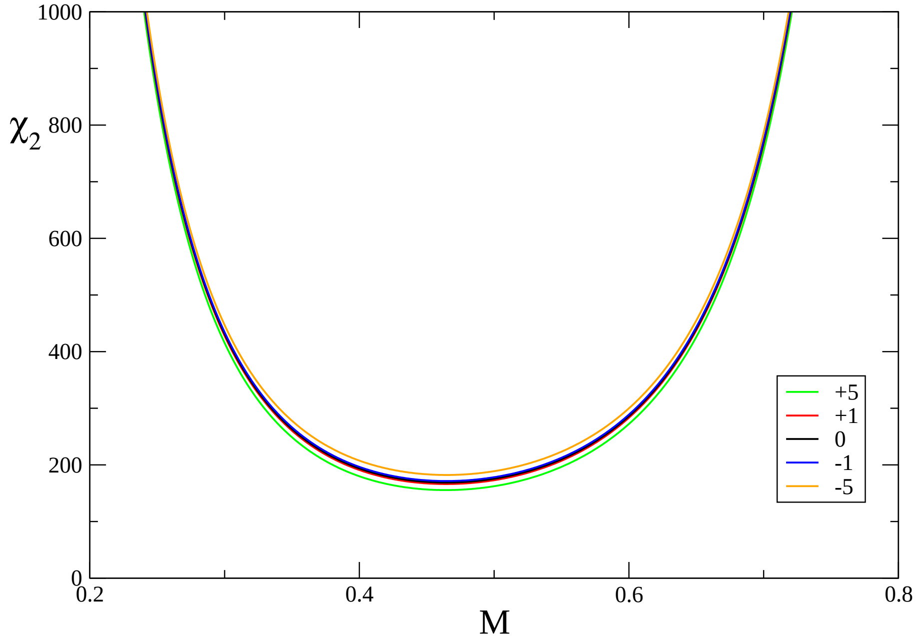


Figure 3: The approximate expression of the NNLO contribution to the BFKL kernel for the unintegrated distribution eq. (4.46) in the $Q_0$ scheme, symmetric variables, $\alpha_s(Q^2)$. See appendix D for the values of all coefficients. The uncertainty band is obtained varying the unknown scheme—fixing coefficient $-5 \leq \dot{g}_{1,-1} \leq 5$ (top to bottom).

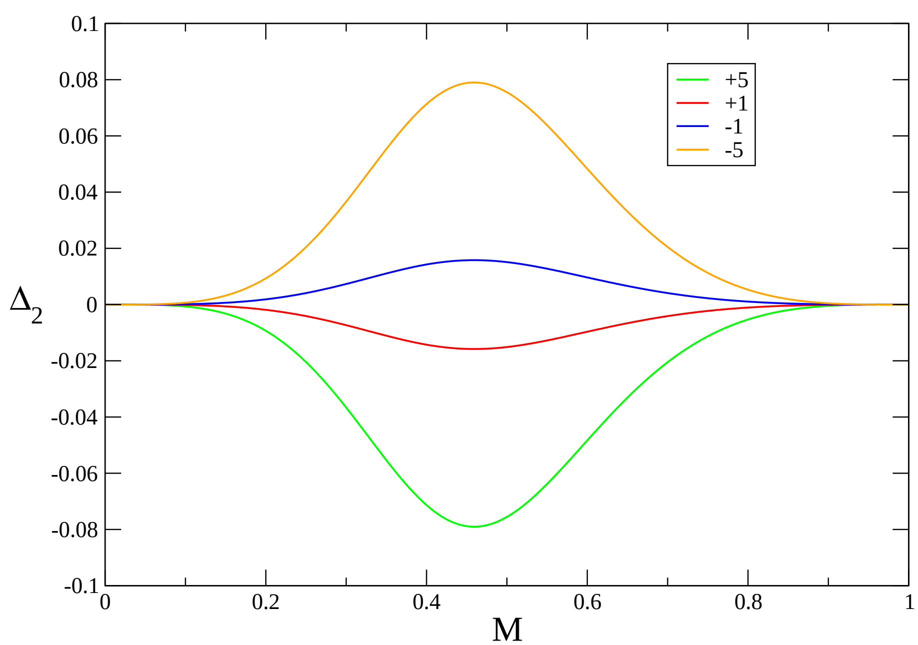


Figure 4: The relative uncertainty in the NNLO contribution to the BFKL kernel for the unintegrated distribution shown in figure 3, due to the uncertainty in the unknown scheme—fixing coefficient $\dot{g}_{1,-1}$, here taking the values $-5, -1, 1, 5$ (top to bottom).

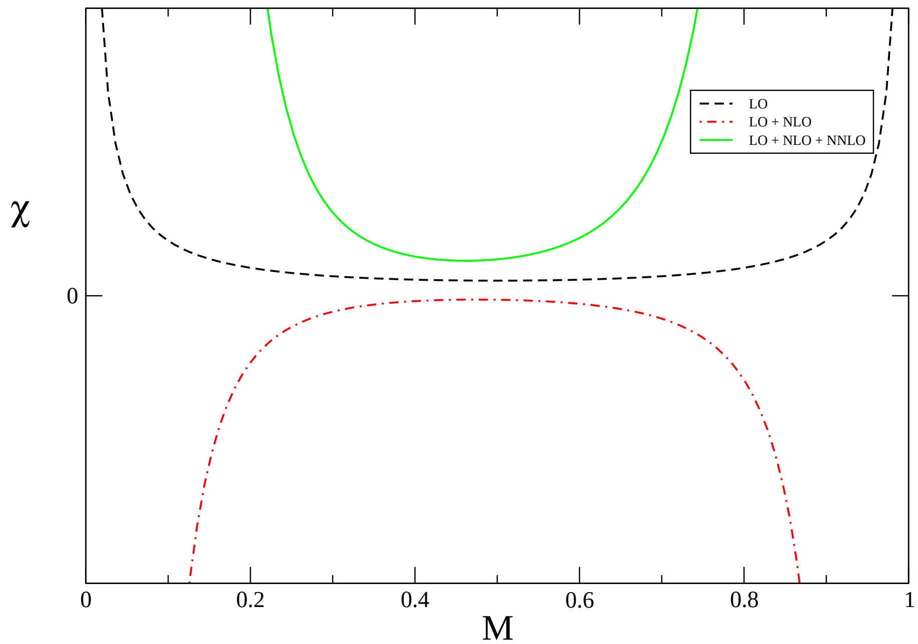


Figure 5: The full BFKL kernel at leading, next-to-leading, and next-to-next-to leading order obtained combining tyhe known expressions for the LO and NLO contributions (as shown in figure 2) and our approximate expression (with $\dot{g}_{1,-1} = 0$) for NNLO (as shown in figure 3), with $\hat{\alpha}_s \to 0.2$. All expressions are in the $\mathrm{Q}_0$ factorization scheme, at the unintegrated level with symmetric variables, and $\alpha_s = \alpha_s(Q^2)$. The symmetry about $M = \frac{1}{2}$ is only broken by the argument of the running coupling.

expansion to coincide with that of $\chi_1^{sym}(M)$ up to $O(M^0)$. No addition is necessary for $\chi_2$ because the known coefficients in its expansion about $M = 0$ are all singular.

Summarizing, all singular coefficients in eqs. (4.41)-(4.43) can be read off eq. (4.36)-(4.38), while for the nonsingular ones we get

$$\begin{aligned} &c_{0,0} = -\tfrac{3}{2} g_{0,-1}, \qquad c_{0,1} = 0, \qquad c_{0,2} = \tfrac{1}{2} g_{0,-1}, \\ &c_{1,0} = \tfrac{g_{2,-2}}{g_{0,-1}} + (g_{0,-1})^2 \zeta(3), \qquad c_{1,1} = \tfrac{1}{2}(g_{0,-1})^2 - g_{0,-1} g_{0,0} - g_{1,-1}. \end{aligned} \tag{4.47}$$

Using these results in eqs. (4.44)-(4.46) we get our approximate expression for the BFKL kernel up to NNLO, at the unintegrated level in the $\mathrm{Q}_0$ scheme in symmetric variables, with $\alpha_s = \alpha_s(Q^2)$. The LO kernel of course does not depend on either scheme, the choice of variables, or the running of the coupling. The NLO kernel corresponds to the widely used form of the kernel as given in ref. [12], eq. (14) of that reference. Indeed, it can be straightforwardly checked that the Laurent expansion of eq. (4.45) coincides with the result of ref. [12] up to and including $O(M^0)$. The NNLO kernel eq. (4.46) is a new result. The expressions for the kernel in DIS variables can be obtained straightforwardly from eqs. (4.44)-(4.46) by inverting eqs. (4.34)-(4.35), and the kernel at the integrated level is found using eq. (4.23). The $\overline{\mathrm{MS}}$ scheme expressions are found using eqs. (4.20)-(4.21) in eq. (4.18).

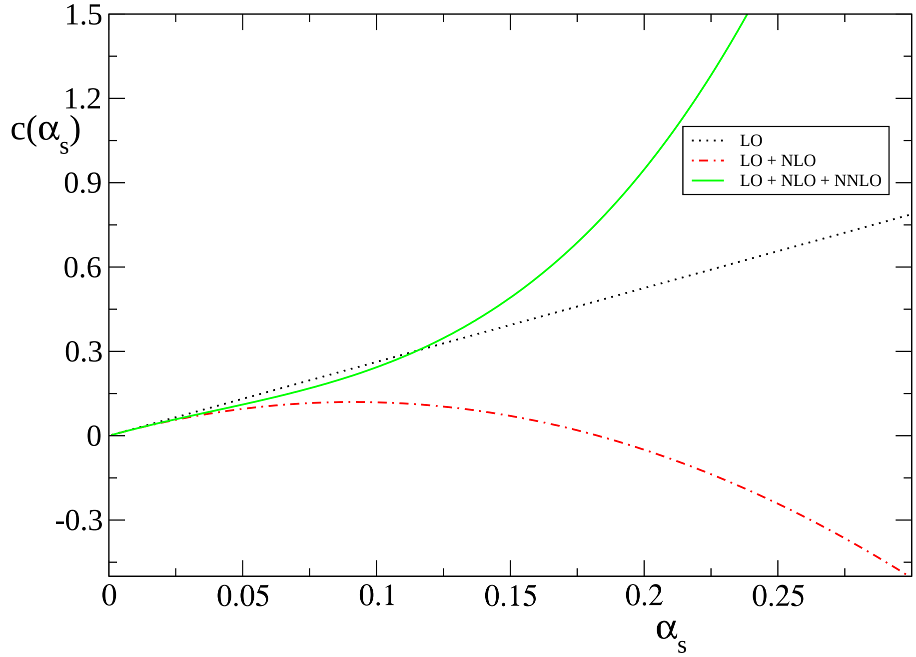


Figure 6: The BFKL intercept $c(\alpha_s)$ at leading, next-to-leading, and next-to-next-to-leading order, obtained from the $\chi$ as shown in figure 5, with a symmetric argument for the running coupling, and evaluated at $M = \frac{1}{2}$.

The full expression for the kernel up to NNLO in the $Q_0$ scheme is given in Appendix D, both in symmetric and DIS variables, at the unintegrated and integrated level. The approximate expressions of the LO and NLO kernels compared to the exact kernels in figure 1 correspond to the expressions eq. (4.44)-(4.45) ($Q_0$ scheme, unintegrated, symmetric variables, $\alpha_s(Q^2)$). The approximate expression of $\chi_2$ is displayed in figure 3, in the $Q_0$ scheme, in symmetric variables, at the unintegrated level and with the canonical argument of the coupling $\alpha_s(Q^2)$. As discussed above, eq. (4.14) (see also Appendix C) one of the coefficients which determine the next-to-next-to-leading order scheme change between the $\overline{\text{MS}}$ and $\overline{\text{MS}}^*$ (and thus $Q_0$), namely $\dot{g}_{1,-1}$, is unknown. This coefficient affects the simple (sub-subleading) poles, and has therefore a moderate impact. Noting that all the scheme-change coefficients (see Appendix C, eq. (C.7)) are of order one or smaller, and indeed all coefficients $g_{i,j}$ (see Appendix B, eq. (B.2)) are at most of order of a few, we estimate the uncertainty related to this coefficient by varying $-5 \leq \dot{g}_{1,-1} \leq 5$. The corresponding uncertainty is displayed in figure 3, and the relative uncertainty in figure 4: it is seen to be similar to the uncertainty of a few percent that we expect (on the basis of the LO and NLO results of figure 1) to affect our approximate form of $\chi_2$.

In figure 5 we display the full NNLO BFKL kernel $\chi(\hat{\alpha}_s, M) = \hat{\alpha}_s\chi_0(M) + \hat{\alpha}_s^2\chi_1(M) + \hat{\alpha}_s^3\chi_2(M)$, with the same scheme and variable choices using the exact expressions up to NLO and the approximate expression to NNLO, with $\hat{\alpha}_s \to 0.2$. The slow convergence properties of the expansion of the BFKL kernel, driven by the increasingly dominant collinear and anti-collinear singularities at $M = 0$ and $M = 1$, are very apparent in this figure. Although the $\chi_2$ contribution restores the minimum near $M = \frac{1}{2}$, the convergence of the expansion in the vicinity of the minimum is still rather slow.

This is made more explicit in figure 6, where we plot the intercept $c(\alpha_s)$ as a function of $\alpha_s$. This is calculated by using the same expression for $\chi$ as in figure 5, but with the coupling chosen to be symmetric, so that all the curves are symmetric about $M = \frac{1}{2}$, and then defining $c(\alpha_s) = \chi(\alpha_s, \frac{1}{2})$, ie the value of $\chi$ the stationary point. While the fixed order perturbation series is clearly good for very small $\alpha_s$, say $\alpha_s \lesssim 0.05$, with $\alpha_s \sim 0.1$ there are signs that the series has yet to converge. For yet larger values of $\alpha_s$ (as would be appropriate for phenomenological studies) the results from the fixed order series are clearly not very useful, and a resummation of collinear and anti-collinear singularities along the lines discussed in Ref. [33,9,22,8] becomes necessary.

## 5. Outlook

In this paper we have presented an approximate determination of the NNLO contribution to the BFKL kernel. In the process, we have provided a full treatment to this order of various issues which affect the determination of the BFKL kernel: the relation between the $\overline{\mathrm{MS}}$ and $\mathrm{Q}_0$ factorization schemes, the duality relations which connect the BFKL kernel to the GLAP anomalous dimension, specifically in the presence of running coupling, the choice of kinematic variables in the definition of the BFKL kernel, the relation between the form of the BFKL kernel and the argument of the running coupling, and the relation between BFKL kernels for integrated and unintegrated parton distributions. All these issues become rather nontrivial to next-to-next-to leading order, and require full control of factorization scheme and running coupling.

Because the perturbative expansion of the BFKL kernel in both the collinear and anticollinear regions is alternating in sign, a knowledge of NNLO corrections is necessary for an accurate assessment of the uncertainty involved in a fixed–order determination of the kernel: indeed, whereas the qualitative features of the NLO kernel are completely different from those of the LO, the NNLO result is qualitatively similar, though we have shown that it is quantitatively not so reliable because of the slow convergence of the perturbative expansion, even in the central region away from the singularities.

Fixed–order BFKL kernels have been widely used recently in studies of nonlinear (saturation) corrections to the BFKL equation and their phenomenological implications for RHIC and the LHC (see e.g. [34] and ref. therein). Also, they are the foundation of numerical approaches to the BFKL equation (see [35] and ref. therein), which in turn are relevant for Monte Carlo simulations (see e.g. ref. [36] and ref. therein). Because of the slow convergence of the perturbative expansion, the determination of the NNLO BFKL kernel presented here is useful in assessing the reliability of these calculations. Finally, the approximate form of the LO and NLO kernels given here are extremely accurate while having a very simple analytic form, and are thus amenable to simple numerical and phenomenological implementations.

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## Appendix A. Higher–order duality

Higher-order duality relations can be obtained by pursuing to higher orders the expansion of the Baker-Campbell-Hausdorff equation for a pair of non-commuting operators $\hat p$, $\hat q$ which act in the same way on a state $G$, according to eq. (2.12). To fifth order we get

$$\begin{aligned} f(\hat q)G(N,M) = \Big[ & f(\hat p) - \tfrac{1}{2} f(\hat p)'' [\hat p,\hat q] + \tfrac{1}{6} f(\hat p)''' [\hat q,[\hat q,\hat p]] + \\ & + \tfrac{1}{3} f(\hat p)''' [\hat p,[\hat p,\hat q]] + \tfrac{1}{24} f(\hat p)^{IV} [\hat q,[\hat q,[\hat q,\hat p]]] + \\ & + \tfrac{1}{8} f(\hat p)^{IV} [\hat q,[\hat p,[\hat p,\hat q]]] - \tfrac{1}{8} f(\hat p)^{IV} [\hat p,[\hat p,[\hat p,\hat q]]] + \\ & + \tfrac{1}{8} f(\hat p)^{IV} [\hat p,\hat q]^2 - \tfrac{1}{24} f(\hat p)^{V} [\hat p,\hat q]\,[\hat q,[\hat q,\hat p]] + \\ & - \tfrac{1}{24} f(\hat p)^{V} [\hat q,[\hat q,\hat p]]\,[\hat p,\hat q] - \tfrac{1}{12} f(\hat p)^{V} [\hat p,\hat q]\,[\hat p,[\hat p,\hat q]] + \\ & - \tfrac{1}{12} f(\hat p)^{V} [\hat p,[\hat p,\hat q]]\,[\hat p,\hat q] - \tfrac{1}{48} f(\hat p)^{VI} [\hat p,\hat q]^3 + O(\hat\alpha_s^4) \Big] G(N,M)\ . \end{aligned} \tag{A.1}$$

We can now obtain higher-order generalizations of the BFKL-like equation obtained starting from a GLAP equation by a suitable identification of $\hat p$ and $\hat q$. Because, as well-known (see e.g. [7]) at the fixed-coupling level the dual of the expansion of $\chi(\alpha_s, M)$ in powers of $\alpha_s$ at fixed $M$ is the expansion of $\gamma(\alpha_s, N)$ in powers of $\alpha_s$ at fixed $\alpha_s/N$, it is convenient to expand

$$\gamma(\hat\alpha_s, N\hat\alpha_s^{-1}) = \gamma_s(N\hat\alpha_s^{-1}) + \hat\alpha_s \gamma_{ss}(N\hat\alpha_s^{-1}) + \ldots \tag{A.2}$$

We then perform the identification eq. (2.16) of $\hat p$ and $\hat q$ and use eq. (A.1) with $f(\hat p) = \bar\chi(\hat\alpha_s, \hat p)$, where

$$\bar\chi(\hat\alpha, \gamma(\hat\alpha, N\hat\alpha^{-1})) = N\hat\alpha^{-1}. \tag{A.3}$$

Namely, $\bar\chi(\hat\alpha_s, M) = \hat\alpha_s^{-1}\tilde\chi(\hat\alpha_s, M)$, where $\tilde\chi$ is the naive (fixed-coupling) dual of $\gamma$, so that

$$\bar\chi(\hat\alpha, M) = \tilde\chi_0(M) + \hat\alpha\tilde\chi_1(M) + \hat\alpha^2\tilde\chi_2(M) + \ldots . \tag{A.4}$$

Using eq. (A.1) we then get

$$\begin{aligned} N\hat\alpha^{-1}G(N,M) = \Big\{ & \bar\chi(\hat\alpha, M) - \tfrac{1}{2}[M,\hat\gamma]\bar\chi''(\hat\alpha, M) - \frac{1}{6}[M,[M,\hat\gamma]]\bar\chi'''(\hat\alpha, M) + \\ & + \frac{1}{6}[\hat\gamma,[\hat\gamma, M]]\bar\chi'''(\hat\alpha, M) + \tfrac{1}{8}[M,\hat\gamma]^2\bar\chi^{IV}(\hat\alpha, M)\Big\} G(N,M)\,, \end{aligned} \tag{A.5}$$

where primes denote derivatives with respect to $M$, and we define $\hat\gamma \equiv \gamma(\hat\alpha_s, N\hat\alpha_s^{-1})$.

All commutators can now be determined explicitly using the expression

$$\hat\alpha_s^{-1} = \frac{1}{\alpha_s} - \beta_0\frac{\partial}{\partial M} + \beta_1\left(-\alpha_s\beta_0\frac{\partial}{\partial M} - \tfrac{1}{2}(\alpha_s\beta_0)^2\frac{\partial^2}{\partial M^2}\right) + O(\alpha_s^3) \tag{A.6}$$

of the running coupling at the operator level:

$$[\hat{\gamma}, M] = -(N\beta_0 + N\beta_0\hat{\alpha}_s\beta_1)\frac{\partial\gamma(\hat{\alpha}_s, N\hat{\alpha}_s^{-1})}{\partial N\hat{\alpha}_s^{-1}} + \\ + \beta_0\hat{\alpha}_s^2\frac{\partial\gamma(\hat{\alpha}_s, N\hat{\alpha}_s^{-1})}{\partial\hat{\alpha}_s} + O(\hat{\alpha}_s^3), \tag{A.7}$$

$$[\hat{\gamma}, [\hat{\gamma}, M]] = O(\hat{\alpha}_s^3) \tag{A.8}$$

$$[M, [M, \hat{\gamma}]] = (N\beta_0)^2\frac{\partial^2\gamma(\hat{\alpha}_s, N\hat{\alpha}_s^{-1})}{\partial(N\hat{\alpha}_s^{-1})^2} + O(\hat{\alpha}_s^3), \tag{A.9}$$

$$([\hat{\gamma}, M])^2 = (N\beta_0)^2\left(\frac{\partial\gamma(\hat{\alpha}_s, N\hat{\alpha}_s^{-1})}{\partial N\hat{\alpha}_s^{-1}}\right)^2 + O(\hat{\alpha}_s^3). \tag{A.10}$$

Substituting the commutators (A.7)-(A.10) in eq. (A.5) and back-substituting order by order the low-order expansion of the enduing equation in the higher-order terms one may remove all the $N$ dependence from the right-hand side of eq. (A.5), with the result:

$$\begin{aligned} N\hat{\alpha}_s^{-1}G = \Big\{&\bar{\chi}(\hat{\alpha}_s, M) - \tfrac{1}{2}\hat{\alpha}_s\beta_0\frac{\bar{\chi}(\hat{\alpha}_s, M)\bar{\chi}''(\hat{\alpha}_s, M)}{\bar{\chi}'(\hat{\alpha}_s, \hat{\gamma})} + \\ &+ \hat{\alpha}_s^2\Big[\tfrac{1}{4}{\beta_0}^2\bar{\chi}(\hat{\alpha}_s, \hat{\gamma})\frac{\bar{\chi}''(\hat{\alpha}_s, \hat{\gamma})^2}{\bar{\chi}'(\hat{\alpha}_s, \hat{\gamma})} - \tfrac{1}{2}\beta_0\beta_1\frac{\bar{\chi}(\hat{\alpha}_s, M)\bar{\chi}''(\hat{\alpha}_s, M)}{\bar{\chi}'(\hat{\alpha}_s, \hat{\gamma})} + \\ &+ {\beta_0}^2\frac{\bar{\chi}(\hat{\alpha}_s, M)^2}{24\bar{\chi}'(\hat{\alpha}_s, M)^4}\Big(12(\bar{\chi}''(\hat{\alpha}_s, M))^3 + \\ &- 7\bar{\chi}'(\hat{\alpha}_s, M)\bar{\chi}''(\hat{\alpha}_s, M)\bar{\chi}'''(\hat{\alpha}_s, M) + 3(\bar{\chi}'(\hat{\alpha}_s, M))^2\bar{\chi}^{IV}(\hat{\alpha}_s, M)\Big) + \\ &- \tfrac{1}{2}\beta_0\frac{\partial\bar{\chi}(\hat{\alpha}_s, M)}{\partial\hat{\alpha}_s}\bar{\chi}''(\hat{\alpha}_s, M)\Big]\Big\}G\,. \end{aligned} \tag{A.11}$$

Identifying the term in curly brackets in eq. (A.11) with the BFKL kernel

$$\chi(\hat{\alpha}_s, M) = \hat{\alpha}_s\chi_0(M) + \hat{\alpha}_s\chi_1(M) + \hat{\alpha}_s^2\chi_2(M) + \dots. \tag{A.12}$$

and expanding $\bar{\chi}$ as in eq. (A.4), eq (A.11) gives an order-by-order expression of the running–coupling dual in terms of the naive dual. Up to NNLO we get

$$\chi_0 = \tilde{\chi}_0 \tag{A.13}$$

$$\chi_1 = \tilde{\chi}_1 - \tfrac{1}{2}\beta_0\frac{\tilde{\chi}_0\tilde{\chi}_0''}{\tilde{\chi}_0'} \tag{A.14}$$

$$\begin{aligned} \chi_2 = \tilde{\chi}_2 &- \tfrac{1}{2}\beta_0\beta_1\frac{\tilde{\chi}_0\tilde{\chi}_0''}{\tilde{\chi}_0'} + \\ &+ \frac{1}{24}\beta_0^2\frac{(\tilde{\chi}_0)^2}{(\tilde{\chi}_0')^4}\left(12\,(\tilde{\chi}_0'')^3 - 14\tilde{\chi}_0'\tilde{\chi}_0''\tilde{\chi}_0''' + 3\,(\tilde{\chi}_0')^2\,\tilde{\chi}^{IV}{}_0\right) + \\ &- \tfrac{1}{2}\beta_0\frac{\tilde{\chi}_0\tilde{\chi}_1''}{\tilde{\chi}_0'} - \beta_0\frac{\tilde{\chi}_1\tilde{\chi}_0''}{\tilde{\chi}_0'} + \tfrac{1}{2}\beta_0\frac{\tilde{\chi}_0\tilde{\chi}_0''\tilde{\chi}_1'}{(\tilde{\chi}_0')^2} \\ &+ \tfrac{1}{4}\beta_0^2\frac{\tilde{\chi}_0}{(\tilde{\chi}_0')^2}\left(2\tilde{\chi}_0'\tilde{\chi}_0''' - (\tilde{\chi}_0'')^2\right), \end{aligned} \tag{A.15}$$

where all $\chi_i$ and $\tilde{\chi}_i$ are functions of $M$ and the prime denotes differentiation with respect to $M$.

By inverting this relation (i.e. expressing order by order $\tilde{\chi}$ in terms of $\chi$), this result can be used to determine the running–coupling corrections to the anomalous dimension $\gamma$ determined from a given BFKL kernel $\chi$. These corrections up to NNLO were first derived in ref. [14], and then reproduced more recently in refs. [18],[7]. However in both cases only the leading-order kernel $\chi_0$ and leading-order running of the coupling were included. For completeness, we derive them here consistently including the running of the coupling up to NLO.

Namely, we start from a given BFKL kernel $\chi(\hat{\alpha}_s, N)$ and determine first the GLAP anomalous dimension $\tilde{\gamma}(\alpha_s, N\alpha_s^{-1})$ which is obtained from it using naive duality, which we then expand according to eq. (A.2). We wish to determine the anomalous dimension $\gamma(\hat{\alpha}_s, N\hat{\alpha}_s^{-1})$ which is related through running–coupling duality to the starting $\chi$. We observe that the naive-duality relations between $\chi$ and $\tilde{\gamma}$ and between $\tilde{\chi}$ and $\gamma$ imply that

$$N/\alpha_s = \chi(\alpha_s, \tilde{\gamma}(\alpha_s, N/\alpha_s)) = \tilde{\chi}(\alpha_s, \gamma(\alpha_s, N/\alpha_s)). \tag{A.16}$$

Note that in eq. (A.16) it is immaterial whether $\alpha_s$ is considered to be an operator or not, because $[\hat{\alpha}_s, N] = 0$ anyway. Defining

$$\begin{aligned} \Delta\chi(\hat{\alpha}_s, M) &= \tilde{\chi}(\hat{\alpha}_s, M) - \chi(\hat{\alpha}_s, M) \\ \Delta\gamma(\hat{\alpha}_s, N\hat{\alpha}_s^{-1}) &= \tilde{\gamma}(\hat{\alpha}_s, N\hat{\alpha}_s^{-1}) - \gamma(\hat{\alpha}_s, N\hat{\alpha}_s^{-1}), \end{aligned} \tag{A.17}$$

and expanding eq. (A.16) in powers of $\alpha_s$ we get

$$\begin{aligned} \Delta\chi_1 &= -\chi_0'\Delta\gamma_{ss} \\ \Delta\chi_2 &= -\chi_0'\Delta\gamma_{sss} - \chi_1\Delta\gamma_{ss}' - \chi_1'\Delta\gamma_{ss} + \tfrac{1}{2}\chi_0''\Delta\gamma_{ss}{}^2 + \chi_0'\Delta\gamma_{ss}'\Delta\gamma_{ss}, \end{aligned} \tag{A.18}$$

where $\Delta\chi_i(M)$ and $\Delta\gamma_i(N\hat{\alpha}_s^{-1})$ are respectively the coefficients of the expansion of $\Delta\chi(\hat{\alpha}_s, M)$ and $\Delta\gamma(\hat{\alpha}_s, N\hat{\alpha}_s^{-1})$ in powers of $\hat{\alpha}_s$, and on the left-hand side all $\Delta\chi_i$ are evaluated as functions $\Delta\chi_i(\tilde{\gamma}(\alpha_s, N\alpha_s^{-1}))$.

Furthermore, inverting eq. (A.15) we can determine the coefficients $\Delta\chi_i$ of the expansion of $\Delta\chi(\alpha_s, M)$ in powers of $\alpha_s$ in terms of $\chi$ (all $\chi_i$ functions of $M$):

$$\begin{aligned} \Delta\chi_1 &= \tfrac{1}{2}\beta_0\frac{\chi_0\chi_0''}{\chi_0'} \\ \Delta\chi_2 &= \tfrac{1}{2}\beta_0\beta_1\frac{\chi_0\chi_0''}{\chi_0'} + \tfrac{1}{2}\beta_0\frac{\chi_0\chi_1''}{\chi_0'} + \beta_0\frac{\chi_1\chi_0''}{\chi_0'} - \tfrac{1}{2}\beta_0\frac{\chi_0\chi_0''\chi_1'}{(\chi_0')^2} \\ &\quad + \frac{1}{24}\beta_0^2\frac{(\chi_0)^2}{(\chi_0')^4}\left(6\,(\chi_0'')^3 - 10\chi_0'\chi_0''\chi_0''' + 3\,(\chi_0')^2\,\chi_0{}^{IV}\right) + \tfrac{1}{4}\beta_0^2\frac{\chi_0}{(\chi_0')^2}\chi_0''^{\,2}. \end{aligned} \tag{A.19}$$

Equating the right-hand side of each of eqs. (A.18) to the corresponding equations (A.19) we determine finally

$$\Delta\gamma_{ss} = -\tfrac{1}{2}\beta_0\frac{\chi_0\chi_0''}{\chi_0'^2}\Big|_{M=\tilde{\gamma}_s} \tag{A.20}$$

and

$$\Delta\gamma_{sss} = \Delta\gamma_{sss}^{(0)} + \Delta\gamma_{sss}^{(1)} + \beta_1 \Delta\gamma_{ss}, \tag{A.21}$$

where we have defined

$$\Delta\gamma_{sss}^{(0)} = -\frac{1}{24}\beta_0^2 \frac{(\chi_0)^2}{(\chi_0')^5}\left(15\left(\chi_0''\right)^3 - 16\chi_0'\chi_0''\chi_0''' + 3\left(\chi_0'\right)^2 \chi_0{}^{IV}\right)\Big|_{M=\tilde{\gamma}_s} \tag{A.22}$$

and

$$\Delta\gamma_{sss}^{(1)} = \left(\tfrac{1}{2}\beta_0\chi_0\chi_0'\tilde{\gamma}_{ss}'' + \tfrac{1}{2}\beta_0\chi_0\chi_1'\tilde{\gamma}_s'' + \tfrac{1}{2}\beta_0\chi_0'\chi_1\tilde{\gamma}_s'' + \tfrac{1}{2}\beta_0\chi_0\chi_0''\tilde{\gamma}_s''\tilde{\gamma}_{ss}\right)\Big|_{M=\tilde{\gamma}_s}. \tag{A.23}$$

The full NNLO running coupling correction eq. (A.21) is given here for the first time. In particular, the term $\Delta\gamma_{sss}^{(0)}$, which depends only on $\chi_0$ and $\beta_0$, was already derived in ref. [14] (eq. (14) of that reference) and later confirmed in ref. [18] [eq. (4.16) of this reference, which gives $\mathcal{N}(N,t)$ eq. (3.11)]. However, the terms $\Delta\gamma_{sss}^{(1)}$, which are due to the next-to-leading order kernel $\chi_1$, and the last term on the right-hand side of eq. (A.20), due to the next-to-leading order running of the coupling were never computed before.

## Appendix B. Expansion of the GLAP anomalous dimensions

The coefficients of the expansion eqs. (4.5)-(4.7) of the leading, next-to-leading and next-to-next-to-leading GLAP anomalous dimensions in powers of $N$ are easily determined by recalling that $\gamma$ is the large eigenvalue of the $2\times 2$ anomalous dimension matrix, given by

$$\gamma = \tfrac{1}{2}\left[\gamma_{gg} + \gamma_{qq} + \sqrt{(\gamma_{gg}-\gamma_{qq})^2 + 4\gamma_{gq}\gamma_{qg}}\right], \tag{B.1}$$

and using the expressions of $\gamma_{ij}$ given in Refs. [31,32,6]. In the $\overline{\rm MS}$ scheme we get

$$\begin{aligned}
g_{0,-1} &= \tfrac{C_A}{\pi}\\
g_{0,0} &= -\tfrac{11C_A}{12\pi} + \left(-\tfrac{1}{6\pi} + \tfrac{C_F}{3\pi C_A}\right)n_f\\
g_{0,1} &= -\tfrac{C_A\pi}{6} + \tfrac{67C_A}{36\pi} - \tfrac{11C_F n_f}{36\pi C_A} + \left(-\tfrac{C_F^2}{9\pi C_A^3} + \tfrac{C_F}{18\pi C_A^2}\right)n_f^2\\
g_{1,-1} &= \left(\tfrac{13C_F}{18\pi^2} - \tfrac{23C_A}{36\pi^2}\right)n_f\\
g_{1,0} &= -\tfrac{2\zeta(3)C_A^2}{\pi^2} + \tfrac{1643C_A^2}{216\pi^2} - \tfrac{11C_A^2}{36} + \left(\tfrac{43C_A}{54\pi^2} + \tfrac{C_F}{18} - \tfrac{547C_F}{216\pi^2} + \tfrac{C_F^2}{4\pi^2 C_A}\right)n_f\\
&\quad + \left(\tfrac{13C_F}{108\pi^2 C_A} - \tfrac{13C_F^2}{54\pi^2 C_A^2}\right)n_f^2\\
g_{2,-2} &= \tfrac{\zeta(3)C_A^3}{2\pi^3} + \tfrac{11C_A^3}{72\pi} - \tfrac{395C_A^3}{108\pi^3} + \left(\tfrac{C_A^2}{36\pi} - \tfrac{71C_A^2}{108\pi^3} - \tfrac{C_FC_A}{18\pi} + \tfrac{71C_FC_A}{54\pi^3}\right)n_f\\
g_{2,-1} &= -\tfrac{143\zeta(3)C_A^3}{24\pi^3} - \tfrac{29\pi C_A^3}{720} - \tfrac{389C_A^3}{432\pi} + \tfrac{73091C_A^3}{2592\pi^3} + \left(-\tfrac{11\zeta(3)C_A^2}{12\pi^3} - \tfrac{C_A^2}{9\pi}\right.\\
&\quad \left. + \tfrac{301C_A^2}{81\pi^3} + \tfrac{8\zeta(3)C_FC_A}{3\pi^3} + \tfrac{35C_FC_A}{108\pi} - \tfrac{28853C_FC_A}{2592\pi^3} - \tfrac{2C_F^2\zeta(3)}{3\pi^3} + \tfrac{11C_F^2}{12\pi^3}\right)n_f\\
&\qquad + \left(\tfrac{59C_A}{648\pi^3} - \tfrac{65C_F}{324\pi^3}\right)n_f^2.
\end{aligned} \tag{B.2}$$

**Appendix C. The $\mathcal{R}$ scheme change to next-to-next-to-leading order.**

The $\mathcal{R}$ scheme change [15,37,18] is related to the fact that the $\overline{\mathrm{MS}}$ anomalous dimension contains interference terms between collinear poles and the $\beta$ function in $d$ dimensions. The factorization of collinear singularities for a $d$–dimensional partonic cross section $\hat{\sigma}$ which depends on a single dimensionful variable $Q^2$ can be written as

$$\sigma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), N, \varepsilon\right) = \sigma^{(0)}(Q^2, \alpha_0, N, \varepsilon) \exp\left[\int_0^{\alpha_s(\mu^2)} d\alpha \frac{\gamma(\alpha, N, \varepsilon)}{\beta(\alpha, \varepsilon)}\right], \tag{C.1}$$

where $\alpha_s(\mu^2)$ is the dimensionless renormalized coupling, $\alpha_0$ is the bare coupling, $\sigma^{(0)}(Q^2, \alpha_0, N, \varepsilon)$ is the regularized cross section and $\sigma(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), N, \varepsilon)$ is free of collinear singularities. The factorization scale is $\mu^2$ and $\gamma(\alpha_s, N, \varepsilon)$ and $\beta(\alpha_s, \varepsilon)$ are respectively the $d$–dimensional anomalous dimension and beta function. The former is defined as the Mellin transform of the $d$–dimensional Altarelli–Parisi splitting function. The latter is given by

$$\beta(\alpha_s, \varepsilon) = \alpha_s \varepsilon + \beta(\alpha_s) \tag{C.2}$$

in terms of the usual four–dimensional $\beta$ function

$$\beta(\alpha_s) = -\beta_0 \alpha_s^2 (1 + \alpha_s \beta_1 + \ldots). \tag{C.3}$$

The $\overline{\mathrm{MS}}$ anomalous dimension is the residue of the simple pole in $\varepsilon$ in the integrand of the exponential in eq. (C.1), namely

$$\begin{aligned} \gamma^{\overline{\mathrm{MS}}}(\alpha_s, N) &= \mathrm{Res}_\varepsilon \left[\frac{\alpha_s \gamma(\alpha_s, N, \varepsilon)}{\beta(\alpha_s, \varepsilon)}\right] \\ &= \gamma(\alpha_s, N) - \frac{\beta(\alpha_s)}{\alpha_s} \dot{\gamma}(\alpha_s, N) + \left(\frac{\beta(\alpha_s)}{\alpha_s}\right)^2 \ddot{\gamma}(\alpha_s, N) + \ldots, \end{aligned} \tag{C.4}$$

where the various coefficients are defined through the Taylor expansion

$$\gamma(\alpha_s, N, \varepsilon) \equiv \gamma(\alpha_s, N) + \varepsilon \dot{\gamma}(\alpha_s, N) + \varepsilon^2 \ddot{\gamma}(N) + \ldots. \tag{C.5}$$

The $\overline{\mathrm{MS}}$ anomalous dimension thus receives two different classes of contributions: pure collinear singularities and interference terms between the $\varepsilon$-dependent anomalous dimension and the poles arising from the expansion of the $d$-dimensional $\beta$-function. In particular up to next-to-next-to leading order we have

$$\begin{aligned} \gamma_0^{\overline{\mathrm{MS}}} &= \gamma_0 \,, \\ \gamma_1^{\overline{\mathrm{MS}}} &= \gamma_1 + \beta_0 \dot{\gamma}_0 \,, \\ \gamma_2^{\overline{\mathrm{MS}}} &= \gamma_2 + \beta_0 \beta_1 \dot{\gamma}_0 + \beta_0^2 \ddot{\gamma}_0 + \beta_0 \dot{\gamma}_1. \end{aligned} \tag{C.6}$$

The $\mathcal{R}$ scheme change, which appears in the relation between $\overline{\mathrm{MS}}$ and $\mathrm{Q}_0$ schemes eq. (3.15), takes us from the $\overline{\mathrm{MS}}$ scheme to the $\overline{\mathrm{MS}}^*$ scheme where the anomalous dimension

is simply given by $\gamma(\alpha_s, N, 0)$, i.e. $\gamma_0(N)$, $\gamma_1(N)$, etc. Thus in order to compute $\gamma^{\overline{\rm MS}^*}$ we have to subtract from the $\overline{\rm MS}$ anomalous dimension eq. (C.6) the contributions coming from the interference between the $d$-dimensional kernel and the $\beta$ function, which are in turn determined from the knowledge of the $d$–dimensional anomalous dimension $\gamma(\alpha_s, N, \varepsilon)$.

The $d$ dimensional leading order splitting functions have been known for a long time, at least for $x < 1$ [38]:

$$
\begin{aligned}
P_{qq}(x,\varepsilon) &= C_F \frac{1}{(1-x)^{\varepsilon}} \left[ \frac{1+x^2}{1-x} - \varepsilon(1-x) \right] + a_{qq}(x,\varepsilon)\delta(1-x)\,, \\
P_{qg}(x,\varepsilon) &= C_F \frac{1}{(1-x)^{\varepsilon}} \left[ \frac{1+(1-x)^2}{x} - \varepsilon x \right]\,, \\
P_{gq}(x,\varepsilon) &= T_R \frac{1}{(1-x)^{\varepsilon}} \left[ 1 - 2x\frac{1-x}{1-\varepsilon} \right]\,, \\
P_{gg}(x,\varepsilon) &= 2C_A \frac{1}{(1-x)^{\varepsilon}} \left[ \frac{x}{1-x} + \frac{1-x}{x} + x(1-x) \right] + a_{gg}(x,\varepsilon)\delta(1-x)\,.
\end{aligned}
\tag{C.7}
$$

The end–point contribution $a_{qq}$ ($a_{gg}$) can be extracted from any process with collinear radiation from incoming quarks, such as Drell-Yan, or gluons, such as Higgs production from gluon fusion: the $O(\alpha_s)$ coefficient of the $\delta(1-x)$ provides a determination of the end–point term in the splitting function after factoring a simple $\varepsilon$ pole and the Born cross section (and a factor of two when there are two incoming partons). Using the known NLO corrections for Drell-Yan [39] and Higgs [40] production we get

$$
\begin{aligned}
a_{qq}(\varepsilon) &= C_F \left[ \frac{2}{\varepsilon} + \frac{3}{2} + \varepsilon\left(4 - \frac{\pi^2}{3}\right) \right] + O(\varepsilon^2)\,, \\
a_{gg}(\varepsilon) &= \frac{2C_A}{\varepsilon} + \frac{11C_A - 4n_f T_R}{6} - \varepsilon\pi^2 + O(\varepsilon^2)\,.
\end{aligned}
\tag{C.8}
$$

The simple $\varepsilon$ pole cancels against that coming from the expansion of $(1-x)^{-(1+\varepsilon)} = \frac{1}{\varepsilon}\delta(1-x) + \ldots$ in the splitting functions $P_{qq}$ and $P_{gg}$, thereby providing a check of the result.

Using the splitting functions (C.7)-(C.8) we can now determine the coefficients of the expansion in powers of $N$ of the large eigenvalue of the $\varepsilon$-dependent GLAP anomalous dimension matrix: we find

$$
\begin{aligned}
\dot\gamma_0(N) &= \frac{\dot g_{0,-2}}{N^2} + \frac{\dot g_{0,-1}}{N} + \dot g_{0,0} + O(N)\,, \\
\ddot\gamma_0(N) &= \frac{\ddot g_{0,-1}}{N} + O(N^0)
\end{aligned}
\tag{C.9}
$$

with

$$
\begin{aligned}
&\dot g_{0,-2} = 0, \qquad \dot g_{0,-1} = 0, \\
&\dot g_{0,0} = -\frac{67}{12\pi} - \frac{7}{81}\frac{n_f}{\pi}, \qquad \ddot g_{0,-1} = -\frac{\pi^2}{12}\,.
\end{aligned}
\tag{C.10}
$$

The next-to-leading order $d$ dimensional splitting function is not available, though in principle it could be extracted from $d$–dimensional splitting amplitudes [41]. However, the $\mathcal{R}$ scheme change has been determined long ago [15] to the level of the next-to-leading $N = 0$ singularities, i.e. for $\gamma_{ss}(\alpha_s/N)$ eq. (3.9). This corresponds to the $O(\varepsilon)$ correction to the leading $N = 0$ singularities, because [18]

$$\begin{aligned}\gamma_s^{\overline{\mathrm{MS}}} &= \gamma_s\,,\\ \gamma_{ss}^{\overline{\mathrm{MS}}} &= \gamma_{ss} + \beta_0\dot\gamma_s,\end{aligned}\tag{C.11}$$

where we have used a notation similar to that of eq. (C.5) but for the expansion eq. (3.9) of the anomalous dimension. Hence, if we let

$$\dot\gamma_1 = \frac{\dot g_{1,-3}}{N^3} + \frac{\dot g_{1,-2}}{N^2} + \frac{\dot g_{1,-1}}{N} + O(N^0)\,. \tag{C.12}$$

the coefficients $\dot g_{1,-3}$ and $\dot g_{1,-2}$ can be extracted using eq. (C.11) from the scheme change of ref. [15].

Equation (C.11) implies that $\gamma_s$ is left unaffected by the scheme change, so it follows from eq. (C.6) that $\dot g_{0,-i} = \dot g_{0,-1} = \dot g_{1,-3} = 0$ thereby confirming the result of eq. (C.10). Also, the scheme change [15] of $\gamma_{ss}$ starts at $O\left(\alpha_s\left(\frac{\alpha_s}{N}\right)^3\right)$

$$\dot\gamma_s\left(\frac{\alpha_s}{N}\right) = 2\zeta(3)\left(\frac{\alpha_s}{N}\right)^3 + O\left(\left(\frac{\alpha_s}{N}\right)^4\right). \tag{C.13}$$

Collecting everything we get

$$\dot g_{1,-3} = 0; \quad \dot g_{1,-2} = 0, \tag{C.14}$$

while the sub–subleading coefficient $\dot g_{1,-1}$ remains undetermined: it would require knowledge of the $O(\varepsilon)$ correction to the simple $N$–pole contribution to $\gamma_1(\alpha_s, N, \varepsilon)$.

Summarizing, the $\overline{\mathrm{MS}}^*$ anomalous dimension is given in terms of the coefficients eq. (B.2) of the expansion of the $\overline{\mathrm{MS}}$ anomalous dimension and of the scheme change coefficients eqs. (C.10),(C.14) by

$$\begin{aligned}\gamma_0^{\overline{\mathrm{MS}}^*} &= \frac{g_{0,-1}}{N} + g_{00} + g_{0,1}N + O(N^2)\,,\\ \gamma_1^{\overline{\mathrm{MS}}^*} &= \frac{g_{1,-1}}{N} + \bar g_{1,0} + O(N)\,,\\ \gamma_2^{\overline{\mathrm{MS}}^*} &= \frac{g_{2,-2}}{N^2} + \frac{\bar g_{2,-1}}{N} + O(N^0)\,,\end{aligned}\tag{C.15}$$

where

$$\begin{aligned}\bar g_{1,0} &= g_{1,0} - \beta_0\dot g_{0,0}\,,\\ \bar g_{2,-1} &= g_{2,-1} - \beta_0\dot g_{1,-1} - \beta_0^2\ddot g_{0,-1}\,.\end{aligned}\tag{C.16}$$

## Appendix D. The BFKL kernel in the $Q_0$ scheme

In this appendix we give explicit expressions for our approximation to the NNLO BFKL kernel in the $Q_0$ factorization scheme. The kernel for evolution of the unintegrated distribution, with the argument of the strong coupling chosen as $\alpha_s(Q^2)$, and the symmetric choice of kinematic variables eq. (4.24) is given by

$$\chi_0(M) = \tfrac{C_A}{\pi}\big(\tfrac{1}{M} + \tfrac{1}{(1-M)} - 1 - M(1-M)\big); \tag{D.1}$$

$$\begin{aligned}
\chi_1(M) = &-\tfrac{C_A^2}{2\pi^2}\big(\tfrac{1}{M^3} + \tfrac{1}{(1-M)^3}\big) + \tfrac{C_A}{\pi}\big(-\tfrac{11C_A}{12\pi} - \tfrac{n_f}{6\pi} + \tfrac{C_F n_f}{3\pi C_A}\big)\big(\tfrac{1}{M^2} + \tfrac{1}{(1-M)^2}\big)\\
&- \tfrac{C_A}{\pi}\beta_0\tfrac{1}{(1-M)^2} + \big(\tfrac{13C_F n_f}{18\pi^2} - \tfrac{23C_A n_f}{36\pi^2}\big)\big(\tfrac{1}{M} + \tfrac{1}{1-M}\big)\\
&+ \big(\tfrac{C_A^2\zeta(3)}{\pi^2} - \tfrac{\zeta(3)C_A^2}{2\pi^2} + \tfrac{11C_A^2}{72} - \tfrac{395C_A^2}{108\pi^2} + \tfrac{C_A n_f}{36} - \tfrac{71C_A n_f}{108\pi^2} - \tfrac{C_F n_f}{18}\\
&+ \tfrac{71C_F n_f}{54\pi^2} + \beta_0\tfrac{C_A}{\pi}\big) + \tfrac{C_A^2}{2\pi^2} - \tfrac{C_A}{\pi}\big(-\tfrac{11C_A}{12\pi} - \tfrac{n_f}{6\pi} + \tfrac{C_F n_f}{3\pi C_A}\big)\\
&- \big(\tfrac{13C_F n_f}{18\pi^2} - \tfrac{23C_A n_f}{36\pi^2}\big) - \tfrac{C_A}{\pi}\beta_0 M;
\end{aligned} \tag{D.2}$$

$$\begin{aligned}
\chi_2(M) = &\tfrac{C_A^3}{2\pi^3}\big(\tfrac{1}{M^5} + \tfrac{1}{(1-M)^5}\big) - \tfrac{C_A^2}{2\pi^2}\big(-\tfrac{11C_A}{4\pi} - \tfrac{n_f}{2\pi} + \tfrac{C_F n_f}{\pi C_A} + \beta_0\big)\big(\tfrac{1}{M^4} + \tfrac{1}{(1-M)^4}\big)\\
&+ 4\beta_0\tfrac{C_A^2}{\pi^2}\tfrac{1}{(1-M)^4} + \tfrac{C_A}{\pi}\Big[\big(-\tfrac{11C_A}{12\pi} - \tfrac{n_f}{6\pi} + \tfrac{C_F n_f}{3\pi C_A}\big)^2\\
&+ \tfrac{C_A}{\pi}\big(-\tfrac{C_F^2 n_f^2}{9C_A^3\pi} + \tfrac{C_F n_f^2}{18C_A^2\pi} - \tfrac{11C_F n_f}{36C_A\pi} - \tfrac{C_A\pi}{6} + \tfrac{67C_A}{36\pi}\big) - \big(\tfrac{13C_F n_f}{18\pi^2} - \tfrac{23C_A n_f}{36\pi^2}\big)\Big]\\
&\cdot\big(\tfrac{1}{M^3} + \tfrac{1}{(1-M)^3}\big) + 2\tfrac{C_A}{\pi}\beta_0\big(\beta_0 + \tfrac{11C_A}{6\pi} + \tfrac{n_f}{3\pi} - \tfrac{2C_F n_f}{3\pi C_A}\big)\tfrac{1}{(1-M)^3}\\
&+ \Big[-\tfrac{1}{2}\big(\tfrac{C_A^3\zeta(3)}{2\pi^3} + \tfrac{11C_A^3}{72\pi} - \tfrac{395C_A^3}{108\pi^3} + \tfrac{C_A^2 n_f}{36\pi} - \tfrac{71C_A^2 n_f}{108\pi^3} - \tfrac{C_A C_F n_f}{18\pi} + \tfrac{71C_A C_F n_f}{54\pi^3}\big)\\
&+ \big(-\tfrac{11C_A}{12\pi} - \tfrac{n_f}{6\pi} + \tfrac{C_F n_f}{3\pi C_A}\big)\big(\tfrac{13C_F n_f}{18\pi^2} - \tfrac{23C_A n_f}{36\pi^2}\big)\\
&+ \tfrac{C_A}{\pi}\big(-\tfrac{2\zeta(3)C_A^2}{\pi^2} + \tfrac{1643C_A^2}{216\pi^2} - \tfrac{11C_A^2}{36} + \tfrac{43C_A n_f}{54\pi^2} + \tfrac{C_F n_f}{18} - \tfrac{547C_F n_f}{216\pi^2} + \tfrac{13C_F n_f^2}{108\pi^2 C_A}\\
&+ \tfrac{C_F^2 n_f}{4\pi^2 C_A} - \tfrac{13C_F^2 n_f^2}{54\pi^2 C_A^2} + \beta_0\big(\tfrac{67}{12\pi} + \tfrac{7n_f}{81\pi}\big)\big) + \tfrac{3\zeta(3)C_A^3}{2\pi^3}\Big]\big(\tfrac{1}{M^2} + \tfrac{1}{(1-M)^2}\big)\\
&- \beta_0\big(\tfrac{C_A}{\pi}\beta_1 + 2\big(\tfrac{13C_F n_f}{18\pi^2} - \tfrac{23C_A n_f}{36\pi^2}\big)\big)\tfrac{1}{(1-M)^2}\\
&+ \Big[-\tfrac{143\zeta(3)C_A^3}{24\pi^3} - \tfrac{29\pi C_A^3}{720} - \tfrac{389C_A^3}{432\pi} + \tfrac{73091C_A^3}{2592\pi^3} - \tfrac{11C_A^2\zeta(3)n_f}{12\pi^3} - \tfrac{C_A^2 n_f}{9\pi}\\
&+ \tfrac{301C_A^2 n_f}{81\pi^3} + \tfrac{8\zeta(3)C_A C_F n_f}{3\pi^3} + \tfrac{35C_A C_F n_f}{108\pi} + \tfrac{59C_A n_f^2}{648\pi^3} - \tfrac{28853C_A C_F n_f}{2592\pi^3}\\
&- \tfrac{2\zeta(3)C_F^2 n_f}{3\pi^3} - \tfrac{65C_F n_f^2}{324\pi^3} + \tfrac{11C_F^2 n_f}{12\pi^3} - \beta_0\dot{g}_{1-1} + \beta_0^2\tfrac{\pi^2}{12} - 2\beta_0\tfrac{\zeta(3)C_A^2}{\pi^2}\Big]\big(\tfrac{1}{M} + \tfrac{1}{1-M}\big).
\end{aligned} \tag{D.3}$$

Substituting the numerical values of the Casimirs $C_A = 3$ and $C_F = \frac{4}{3}$ we get

$$\chi_0(M) = \tfrac{3}{\pi}\big(\tfrac{1}{M} + \tfrac{1}{1-M} - 1 - M(1-M)\big), \tag{D.4}$$

$$\begin{aligned}
\chi_1(M) = &-\tfrac{9}{2\pi^2}\big(\tfrac{1}{M^3} + \tfrac{1}{(1-M)^3}\big) - \big(\tfrac{33}{4\pi^2} + \tfrac{n_f}{18\pi^2}\big)\tfrac{1}{M^2} - \big(\tfrac{33}{2\pi^2} - \tfrac{4n_f}{9\pi^2}\big)\tfrac{1}{(1-M)^2} - \tfrac{103n_f}{108\pi^2}\big(\tfrac{1}{M} + \tfrac{1}{1-M}\big)\\
&+ \tfrac{11}{8} + \tfrac{n_f}{108} - \tfrac{143}{12\pi^2} + \tfrac{47n_f}{162\pi^2} + \tfrac{27\zeta(3)}{2\pi^2} + M\big(-\tfrac{33}{4\pi^2} + \tfrac{n_f}{2\pi^2}\big),
\end{aligned} \tag{D.5}$$

$$
\begin{aligned}
\chi_2(M) = {} & \tfrac{27}{2\pi^3}\big(\tfrac{1}{M^5}+\tfrac{1}{(1-M)^5}\big)+\big(\tfrac{99}{4\pi^3}+\tfrac{n_f}{\pi^3}\big)\tfrac{1}{M^4}+\big(\tfrac{495}{4\pi^3}-\tfrac{5n_f}{\pi^3}\big)\tfrac{1}{(1-M)^4} \\
& +\big[\tfrac{1167}{16\pi^3}+\tfrac{35n_f}{18\pi^3}+\tfrac{n_f^2}{108\pi^3}-\tfrac{9}{2\pi}\big]\tfrac{1}{M^3}+\big[\tfrac{1893}{16\pi^3}+\tfrac{23n_f}{9\pi^3}-\tfrac{7n_f^2}{36\pi^3}-\tfrac{9}{2\pi}\big]\tfrac{1}{(1-M)^3} \\
& +\big[\tfrac{1653}{16\pi^3}+\tfrac{377n_f}{432\pi^3}-\tfrac{5n_f^2}{648\pi^3}+\tfrac{99}{16\pi}+\tfrac{5n_f}{24\pi}-\tfrac{243\zeta(3)}{4\pi^3}\big]\tfrac{1}{M^2} \\
& +\big[\tfrac{1653}{16\pi^3}-\tfrac{5049}{8(33-2n_f)\pi^3}+\tfrac{881n_f}{144\pi^3}+\tfrac{933n_f}{8(33-2n_f)\pi^3}-\tfrac{211n_f^2}{648\pi^3} \\
& -\tfrac{19n_f^2}{4(33-2n_f)\pi^3}+\tfrac{99}{16\pi}+\tfrac{5n_f}{24\pi}-\tfrac{243\zeta(3)}{4\pi^3}\big]\tfrac{1}{(1-M)^2} \\
& +\big[\tfrac{121}{192}-\tfrac{11n_f}{144}+\tfrac{n_f^2}{432}+\tfrac{73091}{96\pi^3}-\tfrac{6125n_f}{648\pi^3}+\tfrac{11n_f^2}{1944\pi^3}-\tfrac{389}{16\pi}-\tfrac{11\dot{g}_{1-1}}{4\pi} \\
& +\tfrac{8n_f}{27\pi}+\tfrac{\dot{g}_{1-1}n_f}{6\pi}-\tfrac{87\pi}{80}-\tfrac{1683\zeta(3)}{8\pi^3}+\tfrac{457n_f\zeta(3)}{108\pi^3}\big]\big(\tfrac{1}{M}+\tfrac{1}{1-M}\big).
\end{aligned}
\tag{D.6}
$$

The expression of the NNLO kernel in DIS variables can be obtained by inverting eqs. (4.30)-(4.31), with the result

$$
\begin{aligned}
\chi_2^{DIS} = {} & \big(-\tfrac{9}{2\pi}+\tfrac{1167}{16\pi^3}-\tfrac{11n_f}{12\pi^3}+\tfrac{n_f^2}{108\pi^3}\big)\tfrac{1}{M^3}+\big(\tfrac{863}{16\pi^3}+\tfrac{33}{4\pi}-\tfrac{54\zeta(3)}{\pi^3}+\tfrac{235n_f}{432\pi^3} \\
& +\tfrac{2n_f}{9\pi}-\tfrac{5n_f^2}{648\pi^3}\big)\tfrac{1}{M^2}+\big(\tfrac{121}{192}+\tfrac{73091}{96\pi^3}-\tfrac{87\pi}{80}-\tfrac{389}{16\pi}-\tfrac{11\dot{g}_{1-1}}{4\pi}-\tfrac{1287\zeta(3)}{8\pi^3} \\
& -\tfrac{11n_f}{144}+\tfrac{8n_f}{27\pi}-\tfrac{6125n_f}{648\pi^3}+\tfrac{\dot{g}_{1-1}n_f}{6\pi}+\tfrac{133n_f\zeta(3)}{108\pi^3}+\tfrac{n_f^2}{432}+\tfrac{11n_f^2}{1944\pi^3}\big)\tfrac{1}{M} \\
& +\tfrac{54}{(1-M)^5\pi^3}+\big(\tfrac{1683}{8\pi^3}-\tfrac{31n_f}{4\pi^3}\big)\tfrac{1}{(1-M)^4} \\
& +\big(-\tfrac{9}{2\pi}+\tfrac{1893}{16\pi^3}+\tfrac{65n_f}{12\pi^3}-\tfrac{7n_f^2}{36\pi^3}\big)\tfrac{1}{(1-M)^3} \\
& +\big(\tfrac{33}{8\pi}+\tfrac{2137}{16\pi^3}-\tfrac{135\zeta(3)}{2\pi^3}+\tfrac{7n_f}{36\pi}+\tfrac{3811n_f}{432\pi^3}-\tfrac{211n_f^2}{648\pi^3}\big)\tfrac{1}{(1-M)^2} \\
& +\big(\tfrac{121}{192}+\tfrac{73091}{96\pi^3}-\tfrac{389}{16\pi}-\tfrac{87\pi}{80}-\tfrac{11\dot{g}_{1-1}}{4\pi}-\tfrac{1287\zeta(3)}{8\pi^3}-\tfrac{11n_f}{144}+\tfrac{8n_f}{27\pi} \\
& -\tfrac{6125n_f}{648\pi^3}+\tfrac{\dot{g}_{1-1}n_f}{6\pi}+\tfrac{133n_f\zeta(3)}{108\pi^3}+\tfrac{11n_f^2}{1944\pi^3}+\tfrac{n_f^2}{432}\big)\tfrac{1}{1-M}.
\end{aligned}
\tag{D.7}
$$

Finally, the kernel for the evolution of the integrated parton density can be obtained from the unintegrated one through eq. (4.23). Using DIS kinematics the difference at NNLO is given by

$$
\begin{aligned}
\chi_2^i(M)-\chi_2^u(M) = {} & \big(-\tfrac{55n_f}{18\pi^3}+\tfrac{5n_f^2}{27\pi^3}\big)\tfrac{1}{M^3}+\tfrac{3(-33+2n_f)}{2\pi^3}\tfrac{1}{(1-M)^3} \\
& +\tfrac{12393-4938n_f+206n_f^2}{648\pi^3}\tfrac{1}{M^2}+\tfrac{-15147+1182n_f-16n_f^2}{108\pi^3}\tfrac{1}{(1-M)^2} \\
& +\tfrac{1}{3888\pi^3}\big[-703890+37974n_f+284n_f^2+29403\pi^2 \\
& -1584n_f\pi^2-12n_f^2\pi^2+96228\zeta(3)-5832n_f\zeta(3)\big]\tfrac{1}{M} \\
& -\tfrac{24543+705n_f-109n_f^2}{324\pi^3}\tfrac{1}{1-M}.
\end{aligned}
\tag{D.8}
$$

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